CP-CONVEXITY AND ITS APPLICATIONS

BY

ICHIRO FUJIMOTO

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

UNIVERSITY OF FLORIDA LIBRARIES

1990

ACKNOWLEDGMENTS

I wish to express my cordial gratitude to Professor Gérard G. Emch for his direction and encouragement throughout my work on this dissertation. I have learned much, not only from his mathematical advice, but also from his sincere attitude directed to complete understanding of fundamental concepts clear from any ambiguity, and his invariable quest for creative research.

I should also be grateful to Professor Nicolae Dinculeanu and Professor Stephen J. Summers for their guidance in related subjects and valuable suggestions which improved my manuscripts.

During the period of my work on this dissertation, I received much advice and encouragement from professors of the supervisory committee, visiting scholars at the University of Florida, researchers in conferences, and from personal communications. Among them, I wish to note my gratitude to Professor Edward G. Effros and Professor Frederic W. Shultz, whose advice were crucial for the improvements of my early manuscripts.

I finally wish to express my thanks to my family and to the Departments of Mathematics, University of Rochester and University of Florida, for their support and hospitality.

TABLE OF CONTENTS

ACKNOWLEDGMENTS ···· ii

ABSTRACT ···· iv

INTRODUCTION ···· 1

CHAPTER 1. CP-CONVEXITY ···· 10

CHAPTER 2. CP-DUALITY FOR C*- AND W*-ALGEBRAS ···· 18

CHAPTER 3. CP-DUALITY FOR JC- AND JW-ALGEBRAS ···· 32

CHAPTER 4. CP-FACIAL STRUCTURES OF CP-STATE SPACES AND DUALITY ···· 48

CHAPTER 5. CP-MEASURE AND INTEGRATION ···· 73

CHAPTER 6. CP-DECOMPOSITION AND CP-CHOQUET THEOREM ···· 92

CHAPTER 7. CP-ORIENTABILITY ···· 108

CHAPTER 8. CP-GEOMETRIC REALIZATION OF TOMITA-TAKESAKI THEORY ···· 131

CHAPTER 9. STONE-WEIERSTRASS THEOREM FOR SEPARABLE C*-ALGEBRAS ···· 152

APPENDIX ···· 173

REFERENCES ···· 188

BIOGRAPHICAL SKETCH ···· 194

Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

CP-CONVEXITY AND ITS APPLICATIONS

By

Ichiro Fujimoto

August 1990

Chairman: Gérard G. Emch
Major Department: Department of Mathematics

We introduce a generalized operator convexity for completely positive maps. which we call "CP-convexity," and develop its theory and applications to operator algebras. This CP-convexity theory can be regarded as a "quantization" of the scalar convexity theory, which interpolates the scalar convexity theory on the state spaces and the algebraic theory for the representations.

Let A be a C*-algebra and H be a Hilbert space such that any cyclic representation of A is realized on H. Then our main stage is the *CP-state space* $Q_H(A)$ of A. which is defined to be the unit ball of the cone $CP(A, B(H))$ of all completely positive maps from A to $B(H)$; a generalization of the quasi-state space of the scalar convexity theory. We define the notions of *CP-convex combination, CP-affine map. CP-face, etc.*, and prove duality theorems between C*- [resp. W*-] algebras and the CP-state spaces [resp. the *normal* CP-state spaces] as the dual objects, and discuss the duality between the ideal structure of the algebra and the CP-facial structure of the CP-state space. This new duality has the advantage to recover the full C*-structure.

The notion of completely positive map and the CP-duality arguments are generalized for JC-algebras, and with these parallel arguments, we formulate simple orientability conditions, which distinguish the category of C*- [resp. W*-] algebras among JC- [resp. JW-] algebras.

We develop a measure and integration theory inherent to CP-convexity, which is a natural generalization of the Lebesgue integral for positive operator map valued measure between order unit spaces, and using this integration theory, we prove decomposition theorems for some classes of CP-maps and generalize the Choquet theorem for CP-state spaces.

We show that we can realize the Tomita-Takesaki theorem in CP-convexity context, and discuss the standard structure in CP-duality. We study the correlation between the CP-orientability and the orientability for the derivations on homogeneous self-dual cones.

Finally, we apply our theory of CP-convexity to prove a generalized Stone-Weierstrass theorem for separable C*-algebras.

INTRODUCTION

In this dissertation, we initiate a theory of operator convexity for completely positive maps and develop its applications to operator algebras. It will turn out through the chapters in this dissertation that this new convexity, which we call "CP-convexity," provides a natural geometric structure for completely positive maps to represent the full C*- or W*-structure of the algebra, in contrast to the classical scalar convexity theory for the state space or the normal state space which essentially describes the order or Jordan structure of the algebra. In brief, our intention in this dissertation is to propose and develop the basic theory of a non-commutative $B(H)$-valued functional analysis for operator algebras, in which the $B(H)$-valued completely bounded maps of the algebra play the role of the dual space, instead of the usual complex valued bounded linear functionals.

In this introduction, we shall give a brief historical sketch of the subjects, from which CP-convexity came out, explain the main ideas of our motivation, and then describe the content of each chapter. The basic notions on operator algebras, completely positive maps, and scalar convexity theory are summarized in the Appendix added at the end of this dissertation.

The traditional scalar convexity theory provides an important point of view in the theory of operator algebras. In this scheme, the self-adjoint part of a unital C*-algebra is treated as an order unit space and the dual space is regarded as the base-norm space with the state space as its base. Then, the self-adjoint part of a unital C*-algebra is order isomorphic to the set of all w*-continuous affine functions on the state space of the algebra, which is known as Kadison's function representation theorem. Since the state space is a w*-compact convex set, it has been an important stage of applications of convexity theory, which is not only of mathematically interest in its own, but also has been studied in connection with the foundations of quantum physics (cf. [32], [33]). The duality between a W*-algebra and its predual has also been studied in parallel (cf. [3] or [12] for convexity theory).

In this duality, a close correlation has been found between the ideal structure of the algebra and the facial structure of the state space or the normal state space of the algebra (cf. E.G. Effros [30] and R.T. Prosser [63]). The study of the duality in this direction was extended by E.M. Alfsen and F.W. Shultz [5] to the non-commutative spectral theory for affine function spaces on convex sets, and this spectral calculus gave rise to the class of JB- and JBW-algebras (cf. [4], [5], [6], [7], [8], [67]). This Alfsen-Shultz theory allows us to understand significant part of the fundamental theory of operator algebras from the point of view of convexity theory, making the order structure of the algebra particularly transparent (cf. also [45] for the structure theory of JB- and JBW-algebras). This approach culminated in the characterization of the state spaces of C*-algebras ([4]) and the normal state space of W*-algebras ([47]); as an important consequence, this allowed to formulate an orientability condition which distinguishes C*- [or W*-] algebras from JB- [or JBW-] algebras. However, these works revealed the difficulty of describing algebraic duality by scalar convexity arguments.

To establish the algebraic duality in more direct manner, we have to seek our dual object in the space of representations. This scheme was initiated by M. Takesaki [73] for separable C*-algebras, and then followed by K. Bichteler [17] for the general case, in which they proved that a C*-algebra is *-isomorphic to the set of all weakly continuous admissible operator valued functions which preserve the direct sum and the unitary equivalence on the space of all representations of the C*-algebra on a Hilbert space H (this space has a sufficiently large dimension so that every cyclic representation is realized on H; see [73] or [17] for the precise statement of the theorem). The proof of the Takesaki duality theorem turned out to be very difficult; this can be ascribed to the complexity of the topological correlation between states and representations.

From these observations, one is naturally led to wish for a duality theory which would preserve algebraic structure as in Takesaki's duality theorem, provide the geometric perspective as in the convexity scheme, and include these two duality theorems. To realize this idea, we propose to generalize the state space to a new dual object which includes both of the state space and the space of representations on a Hilbert space H, and then define a new convexity which interpolates scalar convexity in the state space and the direct sum and unitary equivalence in the representation space. The first claim is realized when we take, as the dual object, the unit ball of all completely positive maps of the C*-algebra A into $B(H)$, where H is the Hilbert space in Takesaki's duality theorem. We call this the *CP-state space* of A and denote it by $Q_H(A)$, which is a generalization of the quasi-state space in the scalar theory. The notion of completely positive map was firstly introduced by W.F. Stinespring [71], and then developed further in its structure theory by W.B. Arveson [11], and now it plays an important role in operator theory and operator algebras and applications (e.g., [33], [34], [55], [56], [59]).

We then propose the following convexity in $Q_H(A)$: $\psi \in Q_H(A)$ is said to be a *CP-convex combination* of $\psi_\alpha \in Q_H(A)$, if there exist $S_\alpha \in B(H)$ with $\sum_\alpha S_\alpha^* S_\alpha \leq I_H$, such that $\psi = \sum_\alpha S_\alpha^* \psi_\alpha S_\alpha$. It is shown that the existence of the above infinite sum is assured from the condition $\sum_\alpha S_\alpha^* S_\alpha \leq I_H$, and that this convexity reduces to the scalar convexity in the case when ψ is one dimensional, i.e., dim ess. supp $(\psi) = 1$; it also describes the direct sum of representations if $\psi_\alpha = \pi_\alpha$ are representations and $S_\alpha = p_\alpha$ are mutually orthogonal projections onto H_{π_α}, and obviously it describes unitary equivalence as a paticular case.

It is interesting to note that a similar expression is found in the theory of operation in C*-algebraic formulation in quantum physics. Assuming that the observables of a physical system are described by the elements of $B(H)$ for some Hilbert space H, K. Kraus [55] showed that the change of the observables caused by an interaction of the system with the exterior is given by a contractive weakly continuous completely positive map in $B(H)$, which is expressed as

$$\psi(a) = \sum_\alpha V_\alpha^* a V_\alpha \quad \text{where } a \in B(H) \text{ and } V_\alpha \in B(H) \text{ such that } \sum_\alpha V_\alpha^* V_\alpha \leq I_H.$$

This expression corresponds to the particular case of our notion of CP-convex combination when we take $\psi_\alpha = id : B(H) \to B(H)$ which is an irreducible representation of $B(H)$. (Unfortunately, he and the successors did not come to perceive the general CP-convexity, and discussed only scalar convexity of the set of such CP-maps.) In the above expression, the coefficients $V_\alpha^* V_\alpha$ are interpreted as the "effect" or the "weight" of the pure operations. Thus, CP-convexity acquires a physical meaning in this particular case, which is expected to develop connections of CP-convexity with applications in the general case. The above generalization of probability measure is developed into CP-measure and integral in Chapter 5.

We may remark here that the motivation given above was not our original motivation. Indeed, our notion of CP-convexity grew out of our study of the general Stone-Weierstrass problem, i.e., a non-commutative generalization of the classical Stone-Weierstrass theorem:

CONJECTURE. *Let A be a unital C^*-algebra and let B be a C^*-subalgebra of A which contains the identity of A, and suppose that B separates the pure states $P(A)$ of A. Then $A = B$.*

In a sense, this generalization of Stone-Weierstrass theorem is in the same direction as the generalization of Gelfand-Naimark theorem, i.e., to reconstruct the original algebra from its dual object. The author has been working for some time on the Stone-Weierstrass problem for the completely positive maps generalizing the pure states to the pure CP-maps ([35], [38]). C.A. Akemann and F.W. Shultz [2] took the same direction in connection with the pure states and the irreducible representations which form a subset of $Q_H(A)$, and studied perfect C*-algebras, which have Stone-Weierstrass property. The idea of our CP-convexity grew out through studying their duality arguments. Our approach to the Stone-Weierstrass problem in CP-convexity will find its result in Chapter 9, where we will prove the general Stone-Weierstrass theorem for separable C*-algebras.

We shall now give brief descriptions of the content of each chapter.

In Chapter 1, we introduce the notion of CP-convexity and related definitions. It will be shown that the CP-state space $Q_H(A)$ of a C*-algebra A is the CP-convex hull of the cyclic representations of A on H, and the quasi-state space in scalar convexity theory is embedded into $Q_H(A)$ as the one dimensional CP-states.

In Chapter 2, we establish the CP-duality between a C*-algebra and the CP-state space, which interpolates Kadison's function representation theorem and Takesaki's duality theorem. We shall show that every *CP-affine* (i.e., preserving the CP-convex combination) BW-homeomorphism between CP-state spaces induces and is induced by a *-isomorphism of the algebras (see Appendix for the definition of BW-topology). This result shows that the Jordan structure of the algebra corresponds to the scalar convexity in the state space, while the C*-structure of the algebra corresponds to the CP-convexity in the CP-state space. We also discuss CP-duality for W*-algebras in parallel.

In Chapter 3, the notion of completely positive map is generalized for JC- and JW-algebras. We first prove the Stinespring representation theorem for JC-algebras, which was made possible by H. Hanche-Olsen's results on tensor product for JC-algebras ([44]). We generalize CP-convexity and CP-duality for JC- and JW-algebras, which is essential for our arguments in Chapters 4, 7 and 8.

In Chapter 4, we discuss CP-affine geometry of CP-state spaces. We define *CP-face* of the CP-state space and study the correlation between the ideals of the algebra and the CP-facial structure of the CP-state space. In this CP-duality, the left, right and two sided ideals are represented geometrically, and more straightforwardly than the scalar theory, without the twisted detour $a \mapsto a^*a$ from the left ideal to the corresponding order ideal. We next discuss CP-facial duality for JC- and JW-algebras, which is a generalization of Alfsen-Shultz's duality in CP-convexity context. From these parallel arguments, one will find that the passage from Jordan structure to C*-structure corresponds to the passage from the scalar convexity on the state space to the CP-convexity on the CP-state space.

Chapter 5 is devoted to develop the measure and integration theory pertinent to CP-convexity, which we are going to apply to formulate a generalization of Choquet's theorem in CP-convexity context in Chapter 6. To understand the situation, observe that a CP-convex combination

$$\psi = \sum_{\alpha} S_\alpha^* \psi_\alpha S_\alpha \quad \text{with } \psi_\alpha \in Q_H(A) \text{ and } S_\alpha \in B(H) \text{ such that } \sum_{\alpha} S_\alpha^* S_\alpha \leq I_H.$$

could be interpreted as a barycenter of the distribution of the effects $S_\alpha^*(\cdot)S_\alpha$, which is considered as a normal CP-state of $B(H)$. We formulate an integral, in strong and weak sense, for more generalized setting with a BW-countably additive measure taking its values in the set of all positive linear maps between order unit spaces. The construction of our integral is straightforward and suitable for operator algebras due to the order unit structure, and has the advantage to have Lebesgue's dominated convergence theorem, which was not realized in the usual method for the general

Banach spaces using semi-variations.

In Chapter 6, we discuss the decomposition problem for completely positive maps and a generalized Choquet theorem for CP-state spaces. We first consider the decomposition into CP-convex sum and show that scattered C*-algebras are characterized by this decomposition property, i.e., every CP-state is decomposed as a CP-convex combination of pure CP-states. We next consider the disintegration of a CP-map into pure elements, and show that this is possible for a class of CP-maps which we call "pre-nuclear" CP-maps. Applying the CP-measure and integration theory developed in Chapter 5, we will show that these decomposition theorems are expressed as CP-boundary measure representation theorems.

Chapter 7 is intended to formulate the orientability condition in the CP-geometrical context. We will formulate the orientability conditions for universally reversible JC-algebras, which are essentially determined by the CP-geometric structure of the CP-state space of the enveloping C*-algebras. We also study the relation between our CP-orientability condition and the 3-ball orientability condition on the state space formulated in [4]. This CP-orientability condition is free from 3-balls, which enables us to discuss C*- and W*-algebras in parallel.

We begin Chapter 8 by showing that every W*-algebra has a faithful normal CP-state, which is the counterpart of a faithful normal semi-finite weight. We use this to define the standard representation by "CP-GNS construction," and generalize Tomita-Takesaki's anti-isomorphism theorem in the CP-geometric context, which was not realized in scalar convexity theory. In this duality, we study the relation between CP-orientability condition and Connes's orientability condition for homogeneous self-dual cones, which is the abstract generalization of the natural self-dual cone of a standard von Neumann algebra. Through these studies, one can see interesting overlaps between Jordan structure and Lie structure in the CP-duality for derivations on the CP-state space with CP-orientability condition. We also discuss *-derivation, *-automorphism, and modular group for JW-algebras.

Our theory of CP-convexity will culminate in Chapter 9 to prove the general Stone-Weierstrass theorem for separable C*-algebras, which was conjectured by S. Sakai [64]. The general Stone-Weierstrass problem has a long history, and has been open even for the separable case. This result provides an example which proves the usefulness of CP-convexity for the study of operator algebras.

CHAPTER 1

CP-CONVEXITY

We introduce a new generalized operator convexity, which we shall call *CP-convexity,* in the cone $CP(A, B(H))$ of all completely positive linear maps from a C*-algebra A into the C*-algebra $B(H)$ of all bounded linear operators on a Hilbert space H.

The definition and basic properties of completely positive maps are given in references, e.g., [11], [59], [71] and [75] (we summarize these in Appendix 2). Recall in particular (Stinespring [71]) that every $\psi \in CP(A, B(H))$ can be written as

$$\psi(a) = V^*\pi(a)V \qquad \text{for all } a \in A,$$

where π is a representation of A on a Hilbert space K and $V \in B(H, K)$ is a bounded linear operator from H to K such that $\|\psi\| = \|V\|^2$. We assume, without loss of generality, the minimal condition $K = [\pi(A)VH]$ (where $[\ ]$ represents the closed linear span) under which the Stinespring representation is unique up to unitary intertwining operators.

Let us define our new concept of convexity in this cone $CP(A, B(H))$.

DEFINITION 1.1. *Let $(\psi_\alpha)_{\alpha\in\Lambda}$ be a bounded family in $CP(A, B(H))$. We define a CP-convex combination of $(\psi_\alpha)_{\alpha\in\Lambda}$ by*

$$\sum_{\alpha\in\Lambda} S_\alpha^*\psi_\alpha S_\alpha \quad \text{with } S_\alpha \in B(H) \text{ such that } \sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha \le I_H,$$

where $\sum_\alpha$ *converges in the BW-topology (i.e., the topology of pointwise weak convergence for bounded nets (cf.[11])). For a bounded set* B *in* $CP(A, B(H))$*, the* ***CP-convex hull*** *of* B*, denoted by* $CP - convB$*, is defined to be the set of all CP-convex combinations of bounded families in* B*. A subset* $K \subset CP(A, B(H))$ *is defined to be* ***CP-convex*** *if it is closed under the operation of CP-convex combination.*

In the above definition, the convergence of the CP-convex combination is secured from the condition $\sum_\alpha S_\alpha^* S_\alpha \leq I_H$. For the completeness of the definition, we shall prove this fact.

PROPOSITION 1.2. *For any bounded family* $(\psi_\alpha)_{\alpha \in \Lambda}$ *in* $CP(A, B(H))$ *and for any family* $(S_\alpha)_{\alpha \in \Lambda}$ *in* $B(H)$ *such that* $\sum_{\alpha \in \Lambda} S_\alpha^* S_\alpha \leq I_H$*, the CP-convex combination* $\sum_{\alpha \in \Lambda} S_\alpha^* \psi_\alpha S_\alpha$ *converges in the BW-topology to an element* $\psi \in CP(A, B(H))$ *with* $\|\psi\| \leq \sup_{\alpha \in \Lambda} \|\psi_\alpha\| \quad (:= r)$.

PROOF: Let $a \in A^+ (:= \{a \in A; a \geq 0\})$ and let $J \subset \Lambda$ be any finite subset. Then,

$$\psi_J(a) := \sum_{\alpha \in J} S_\alpha^* \psi_\alpha(a) S_\alpha \leq \sum_{\alpha \in J} S_\alpha^* (\|\psi_\alpha\| \|a\| I_H) S_\alpha \leq r \|a\| \sum_{\alpha \in J} S_\alpha^* S_\alpha \leq r \|a\| I_H.$$

Thus, $(\psi_J(a))_{J \subset \Lambda}$ (J: finite subsets of Λ) is a bounded increasing net in $B(H)$; hence it converges weakly to an element in $B(H)$. Since every element $a \in A$ is decomposed as a linear combination of positive elements, i.e.,

$$a = a_1 - a_2 + i(a_3 - a_4) \quad \text{with } a_i \in A^+ \ (1 \leq i \leq 4),$$

we have

$$\|\psi_J(a)\| \leq r \sum_{i=1}^{4} \|a_i\| \qquad \text{for all } a \in A,$$

so $(\psi_J)_{J \subset \Lambda}$ is a bounded Cauchy net in the BW-topology. Since a bounded norm-closed ball of $CP(A, B(H))$ is BW-compact (cf. [11; p147]), the net $(\psi_J)_{J \subset \Lambda}$ converges to an element $\psi \in CP(A, B(H))$ in the BW-topology, i.e., $\psi = \lim_J \psi_J = \sum_{\alpha \in \Lambda} S_\alpha^* \psi_\alpha S_\alpha$.

It is left to show $\|\psi\| \leq r$. Note that

$$\|\psi\|^2 = \sup_{\|a\| \leq 1} \|\psi(a)\|^2 = \sup_{\|a\| \leq 1} \|\psi(a)^* \psi(a)\| \leq \|\psi\| \sup_{\|a\| \leq 1} \|\psi(a^* a)\|.$$

where we used the inequality

$$\psi(a)^* \psi(a) \leq \|\psi\| \psi(a^* a) \qquad \text{for all } a \in A$$

(e.g., [75; Ch.IV, Cor. 3.8]). From the first part of the proof, we have

$$\psi(a) = \lim_J \psi_J(a) \leq r\|a\| I_H \qquad \text{for } a \in A^+.$$

Hence, $\sup_{\|a\| \leq 1} \|\psi(a^* a)\| \leq \sup_{\|a\| \leq 1} r\|a^* a\| \leq r$, from which $\|\psi\| \leq r$ follows. ∎

CP-facial structure of CP-convex sets will be discussed in details in Chapter 4. In what follows, throughout this dissertation, we are mainly concerned with the unit ball of the cone $CP(A, B(H))$ for a large Hilbert space H; note that this is CP-convex by Proposition 1.2.

Definition 1.3. *A CP-map $\psi \in CP(A, B(H))$ is called a CP-state if it is a contraction, i.e., $\|\psi\| \leq 1$. We denote by $Q_H(A)$ the set of all CP-states from A to $B(H)$, i.e.,*

$$Q_H(A) = \{\psi \in CP(A, B(H)); \|\psi\| \leq 1\},$$

and we call $Q_H(A)$ *the* ***CP-state space of A for H*** *. Similarly, if* M *is a* W^**-algebra, we define a normal CP-contraction from* M *to* $B(H)$ *to be a* ***normal CP-state***, *and denote by* $Q_H(M)_n$ *the* ***normal CP-state space of M for H*** , i.e.,

$$Q_H(M)_n = \{\psi \in CP(M, B(H))_n; \|\psi\| \le 1\}.$$

We define the ***cyclic dimension*** $\alpha_c(A)$ *of a* C^**-algebra* A *by*

$$\alpha_c(A) := \sup\{dim H_\pi; \pi \text{ is a cyclic representation of } A \text{ on } H_\pi\},$$

and similarly the ***normal cyclic dimension*** $\alpha_c(M)_*$ *of a* W^**-algebra* M *by*

$$\alpha_c(M)_n := \sup\{dim H_\pi; \pi \text{ is a normal cyclic representation of } M \text{ on } H_\pi\}.$$

Remark. $Q_H(A)$ [resp. $Q_H(M)_n$] generalizes the quasi-state space $Q(A)$ [resp. the normal quasi-state space $Q(M)_n$] of the scalar theory (where $H = \mathrm{C}$).

The concept of CP-state space will be essential in the discussion of CP-duality for C*- and W*-algebras in Chapter 2. It is also a natural generalization of "operations" in the applications to physics (cf. Introduction).

PROPOSITION 1.4.A. *Let* A *be a* C^**-algebra and* H *be a Hilbert space with* $dim H \ge \alpha_c(A)$. *Then,*

$$Q_H(A) = CP - conv\ Rep_c(A : H)$$

where $Rep_c(A : H)$ *denotes the set of all cyclic representations of* A *on* H.

PROOF: Let $\psi \in Q_H(A)$, and let $\psi = V^*\pi V$ be the Stinespring representation of ψ where π is a representation of A on a Hilbert space K and $V \in B(H, K)$. Since

π is non-degenerate from the minimal condition, π can be decomposed into a direct sum of cyclic representations $(\pi_\alpha)_{\alpha\in\Lambda}$, i.e., $\pi = \oplus_{\alpha\in\Lambda}\pi_\alpha$ ([28; 2.2.7]). Let us denote by p_α the projection of K onto the representation space H_{π_α} of π_α. Then,

$$\psi = V^*\pi V = V^*(\oplus_{\alpha\in\Lambda}p_\alpha)^*(\oplus_{\alpha\in\Lambda}\pi_\alpha)(\oplus_{\alpha\in\Lambda}p_\alpha)V = \sum_{\alpha\in\Lambda} V^*p_\alpha^*\pi_\alpha p_\alpha V.$$

From the assumption $dimH \geq \alpha_c(A)$, there exists a partial isometry $W_\alpha : H \to H_{\pi_\alpha} \subset K$ of H onto H_{π_α} for each $\alpha \in \Lambda$. In this case $W_\alpha W_\alpha^* = p_\alpha$, so we have

$$\psi = \sum_{\alpha\in\Lambda} V^*(W_\alpha W_\alpha^*)^*\pi_\alpha(W_\alpha W_\alpha^*)V = \sum_{\alpha\in\Lambda}(W_\alpha^*V)^*(W_\alpha^*\pi_\alpha W_\alpha)(W_\alpha^*V).$$

Note here that $W_\alpha^*\pi_\alpha W_\alpha \in Rep_c(A : H)$ since W_α is an partial isometry, and $W_\alpha^*V \in B(H)$ satisfies

$$\begin{aligned}\sum_{\alpha\in\Lambda}(W_\alpha^*V)^*(W_\alpha^*V) &= \sum_{\alpha\in\Lambda} V^*(W_\alpha W_\alpha^*)V = \sum_{\alpha\in\Lambda} V^*p_\alpha V \\ &= V^*(\oplus_{\alpha\in\Lambda}p_\alpha)V = V^*V \leq \|V\|^2 I_H \leq I_H.\end{aligned}$$

Thus we proved that ψ is a CP-convex combination of cyclic representations of A on H, i.e., $Q_H(A) \subset CP-conv\ Rep_c(A : H)$. Since the inverse inclusion is obvious, the proposition is proved. ▮

By similar arguments, we can show

PROPOSITION 1.4.B. *Let M be a W^*-algebra, and H be a Hilbert space with $dimH \geq \alpha_c(M)_n$. Then,*

$$Q_H(M)_n = CP - conv\ Rep_c(M : H)_n$$

where $Rep_c(M : H)_n$ denotes the set of all normal cyclic representations of M on H.

In Chapter 6, we will show that, if A is a separable C*-algebra, every pre-nuclear CP-map $\psi \in CP(A, B(H))$ (see Chapter 6 for the definition) is decomposed into pure CP-maps in the sense of weak integral, which is crucial for our arguments of Stone-Weierstrass problem (separable case) in Chapter 9.

The following proposition shows that the quasi-states in the scalar theory are characterized as one dimensional CP-states, which will be used in the discussion of CP-duality for C*- and W*-algrbras in Chapter 2.

PROPOSITION 1.5.A. *Let A be a C*-algebra, and H be a Hilbert space. Then, the quasi-state space $Q(A)$ of A can be regarded as the set of all equivalence classes of one dimensional CP-states of A for H, i.e.,*

$$Q(A) \cong \{\hat{\psi}; \psi = V^*\pi V \in Q_H(A),\ dim\,[im(V)] = 1\} := \hat{Q}_{H,1}(A)$$

where ˆ represents the unitary equivalence class; the correspondence is given by

$$\omega \in Q(A) \quad \longleftrightarrow \quad \hat{\psi_\omega} := (\omega(\cdot)P)\hat{} \in \hat{Q}_{H,1}(A)$$

where P is a one dimensional projection on H.

PROOF: Let $\omega \in Q(A)$. By the GNS-construction, there exists a cyclic representation π_ω of A on H_{π_ω}, and a cyclic vector $\xi_\omega \in H_{\pi_\omega}$ for π_ω such that

$$\omega(a) = (\pi_\omega(a)\xi_\omega, \xi_\omega) \qquad \text{for all } a \in A.$$

We take and fix an arbitrary $h \in H$ with $\|h\| = 1$, and define $V_\omega \in B(H, H_{\pi_\omega})$ by

$$V_\omega k = (k, h)\xi_\omega \qquad \text{for all } k \in H,$$

and define $\psi_{\omega,h} \in CP(A, B(H))$ by

$$\psi_{\omega,h}(a) = V_\omega^* \pi_\omega(a) V_\omega \qquad \text{for all } a \in A.$$

Then, for any $f, g \in H$ and $a \in A$,

$$(\psi_{\omega,h}(a)f, g) = (\pi_\omega(a)V_\omega f, V_\omega g) = (\pi_\omega(a)(f,h)\xi_\omega, (g,h)\xi_\omega)$$

$$= (f,h)(h,g)(\pi_\omega(a)\xi_\omega, \xi_\omega) = (\omega(a)(f,h)h, g) = (\omega(a)P_h f, g)$$

where $P_h := (\cdot, h)h$ is a one dimensional projection of H onto $[h]$. Since $f, g \in H$ are arbitrary, we have

$$\psi_{\omega,h}(a) = \omega(a)P_h \qquad \text{for all } a \in A.$$

Conversely, let $\psi = V^*\pi V \in Q_H(A)$ and $dim\,[im(V)] = 1$. Take $h \in im(V^*)$ with $\|h\| = 1$. Then, $V \in B(H, K)$ must be of the form

$$Vk = (k,h)Vh \qquad \text{for all } k \in H,$$

and the minimal condition $H_\pi = [\pi(A)VH]$ implies that π is a cyclic representation with a cyclic vector $\xi_\pi := Vh$. As we showed in the first part of the proof, we can see

$$\psi = \omega_\psi(\cdot)P_h,$$

where

$$\omega_\psi \in Q(A) \qquad \text{such that} \qquad \omega_\psi(a) = (\pi(a)\xi_\pi, \xi_\pi) \qquad \text{for all } a \in A.$$

By considering the equivalence classes, we get the desired correspondence

$$\omega \in Q(A) \quad \leftrightarrow \quad \hat{\psi}_\omega = (\omega(\cdot)P)\hat{} \in \hat{Q}_{H,1}(A).$$

∎

Similarly, for W*-algebras, we can show the following.

PROPOSITION 1.5.B. *Let M be a W^*-algebra, and H be a Hilbert space. Then. the normal quasi-state space $Q(M)_n$ of M can be regarded as the set of all equivalence classes of one dimensional normal CP-states of M for H, i.e.,*

$$Q(M)_n \cong \{\hat{\psi}; \psi = V^*\pi V \in Q_H(M)_n,\ dim\,[im(V)] = 1\} := \hat{Q}_{H,1}(M)_n$$

where ˆ represents the unitary equivalence class; the correspondence is given by

$$\omega \in Q(M)_n \quad \longleftrightarrow \quad \hat{\psi}_\omega := (\omega(\cdot)P)\hat{} \in \hat{Q}_{H,1}(M)_n$$

where P is a one dimensional projection on H.

Remark. Propositions 1.5.A [resp. B] state that $Q(A)$ [resp. $Q(M)_n$] can be regarded as $Q_{\mathbf{C}}(A)$ [resp.$Q_{\mathbf{C}}(M)_n$].

CHAPTER 2

CP-DUALITY FOR C*-AND W*-ALGEBRAS

In this chapter we establish a duality theory for C*- and W*-algebras in the context of CP-convexity, where we take as our primary object the CP-state space $Q_H(A)$ [resp. the normal CP-state space $Q_H(M)_n$] rather than the quasi-state space $Q(A)$ of a C*-algebra A [resp. the normal state space $Q(M)_n$ of a W*-algebra M] which were used in the scalar convexity theory. We first generalize the notion of affine functions in the classical scalar convexity theory for our CP-convexity context.

DEFINITION 2.1. *Let A be a C*-algebra. A function $\gamma : Q_H(A) \to B(H)$ is defined to be CP-affine, if*

$$\psi = \sum_\alpha S_\alpha^* \psi_\alpha S_\alpha \quad \textit{with } \psi_\alpha \in Q_H(A) \textit{ and } S_\alpha \in B(H) \textit{ such that } \sum_\alpha S_\alpha^* S_\alpha \leq I_H$$

implies that

$$\gamma(\psi) = \sum_\alpha S_\alpha^* \gamma(\psi_\alpha) S_\alpha.$$

*γ is defined to be **bounded** if $\|\gamma\| = \sup\{\|\gamma(\psi)\|; \psi \in Q_H(A)\} < \infty$. We denote by $AB(Q_H(A), B(H))$ the set of all bounded CP-affine functions from $Q_H(A)$ to $B(H)$, and by $AC(Q_H(A), B(H))$ the set of all BW-w continuous CP-affine functions from $Q_H(A)$ to $B(H)$. The notations for W*-algebras are defined similarly.*

The following duality theorems generalize Kadison's function representation theorem ([50]) and have the advantage that they recover the full C*-structure of the original algebra, i.e., C*-product and C*-norm; this structure would have escaped our grasp had we considered only the set of affine functions on the quasi-state spaces as is customarily done in the scalar theory.

THEOREM 2.2.A. *Let A be a C*-algebra and H be a Hilbert space with* $dimH \geq \alpha_c(A)$. *Then,* $AC(Q_H(A), B(H))$ *is a C*-algebra with the following product; for* $\gamma_1, \gamma_2 \in AC(Q_H(A), B(H))$ *and* $\psi = V^*\pi V \in Q_H(A)$ *with CP-decomposition*

$$\psi = \sum_\alpha V_\alpha^* \pi_\alpha V_\alpha \quad \text{where } \pi_\alpha \in Rep_c(A:H) \text{ with } \pi \simeq \oplus_\alpha \pi_\alpha$$
$$\text{and } V_\alpha \in B(H) \text{ such that } \sum_\alpha V_\alpha^* V_\alpha \leq I_H \quad \text{(cf. Prop.1.4.A)},$$

we define the product of γ_1 *and* γ_2 *by*

$$(\gamma_1 \cdot \gamma_2)(\psi) := \sum_\alpha V_\alpha^* \gamma_1(\pi_\alpha) \cdot \gamma_2(\pi_\alpha) V_\alpha,$$

and we have

$$A \cong AC(Q_H(A), B(H)) \qquad \text{(*-isomorphism)}.$$

THEOREM 2.2.B. *Let M be a W*-algebra and H be a Hilbert space with* $dimH \geq \alpha_c(M)_n$. *Then,* $AB(Q_H(M)_n, B(H))$ *is a W*-algebra with respect to the product defined in Proposition 2.2.A, where* $Rep_c(A:H)$ *is replaced by* $Rep_c(M:H)_n$, *and we have*

$$M \cong AB(Q_H(M)_n, B(H)) \qquad \text{(*-isomorphism)}.$$

PROOF: The proofs of the above two theorems proceed in parallel, and Theorem 2.2.A is obtained from Theorem 2.2.B by taking $M = A^{**}$ and considering the

BW-w continuous elements, so that it is sufficient to prove Theorem 2.2.B, which we now establish.

As the first step, we show that there exists a one to one correspondence between M and $AB(Q_H(M)_n, B(H))$.

We recall the fact that the normal quasi-state space $Q(M)_n$ of M can be regarded as the set of all unitary equivalence classes of one dimentional normal CP-states from M to $B(H)$; the correspondence is given by

$$\omega \in Q(M)_n \quad \longleftrightarrow \quad \hat{\psi_\omega} = (\omega(\cdot)P)\hat{} \in \hat{Q}_{H,1}(M)_n$$

where P is a one dimensional projection on H and $\hat{}$ represents the unitary equivalence class (cf. Prop. 1.5.B).

We consider the following diagram:

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "k"'] & AB(Q_H(M)_n, B(H)) \arrow[d, "j"] \\ & A_0^b(Q(M)_n, C) \end{tikzcd}$$

where $A_0^b(Q(M)_n, C)$ denotes the set of all complex valued bounded affine functions on $Q(M)_n$ vanishing at 0, and i assigns the evaluation map at each element of M, i.e.,

$$i(a) : \psi \longmapsto \psi(a) \quad \text{for } a \in M \text{ and } \psi \in Q_H(M)_n,$$

j is defined for $\gamma \in AB(Q_H(M)_n, B(H))$ by

$$j(\gamma)(\omega) = (\gamma(\psi_\omega)h, h) \quad \text{with } h \in P(H), \ \|h\| = 1,$$

and k is the Kadison's isomorphism (see Appendix 1).

We first note that the above diagram commutes, i.e., $j \circ i = k$. Indeed, let $a \in M$ and $\omega \in Q(M)_n$. Then,

$$\begin{aligned} j(i(a))(\omega) &= (i(a)(\psi_\omega)h, h) = (\psi_\omega(a)h, h) \\ &= (\omega(a)Ph, h) = \omega(a) := k(a)(\omega). \end{aligned}$$

(cf. Proof of Prop. 1.5)

It is immediate to see that i is injective, since, if $i(a) = 0$ on $Q_H(M)_n$, then $i(a) = 0$ on $\hat{Q}_{H,1}(M)_n$, hence on $Q(M)_n$ (cf. Prop. 1.5.B), which implies $a = 0$.

We now show that j is injective. Let $\gamma \in AB(Q_H(M)_n, B(H))$ be arbitrary such that $\gamma \neq 0$, and assume $j(\gamma) = 0$. We recall from the proof of Prop. 1.5,

$$\psi_\omega = V_\omega^* \pi_\omega V_\omega \quad \text{where } V_\omega = (\cdot, h)\xi_\omega \text{ with } h \in P(H),\ \|h\| = 1.$$

We note here that V_ω has the polar decomposition $V_\omega = \|\omega\|^{\frac{1}{2}} u_\omega P$ where $u_\omega : H \to H_{\pi_\omega}$ is a partial isometry with $u_\omega^* u_\omega = id_{[h]}$. Then, due to our assumption $dim H \geq \alpha_c(M)_n$, we can extend u_ω to a partial isometry $U_\omega : H \to H_{\pi_\omega}$ from H onto H_{π_ω}, i.e., $U_\omega U_\omega^* = H_{\pi_\omega}$ and $U_\omega|_{[h]} = u_\omega$. Thus

$$\psi_\omega = \|\omega\| P U_\omega^* \pi_\omega U_\omega P \quad \text{where} \quad U_\omega^* \pi_\omega U_\omega \in Rep_c(M : H)_n.$$

Then our assumption $j(\gamma) = 0$ implies

$$\|\omega\|(P\gamma(U_\omega^* \pi_\omega U_\omega)Ph, h) = 0 \quad \text{for all } \omega \in Q(M)_n,$$

i.e.,

$$(\gamma(U_\omega^* \pi_\omega U_\omega)h, h) = 0 \quad \text{for all non-zero } \omega \in Q(M)_n.$$

Noting that $h = U_\omega^* \xi_\omega$ is a cyclic vector for the normal cyclic representation $\tilde{\pi}_\omega = U_\omega^* \pi_\omega U_\omega$, it is equivalent to asserting that

$$(\gamma(\pi)\xi, \xi) = 0 \quad \text{for all } \pi \in Rep_c(M : H)_n \text{ and a cyclic vector } \xi \text{ for } \pi.$$

Indeed, let $\pi \in Rep_c(M : H)_n$ and ξ be a cyclic vector for π, and set $\omega_\pi(\cdot) = (\pi(\cdot)\xi, \xi) \in Q(M)_n$. By the GNS-construction, there exists a representation π_{ω_π} on $H_{\pi_{\omega_\pi}}$, a cyclic vector ξ_{ω_π} for π_{ω_π} such that $\omega_\pi(\cdot) = (\pi_{\omega_\pi}(\cdot)\xi_{\omega_\pi}, \xi_{\omega_\pi})$. Hence, there exists a unitary operator $U_\pi : H_\pi \to H_{\pi_{\omega_\pi}}$ such that $\pi = U_\pi^* \pi_{\omega_\pi} U_\pi$ and $\xi = U_\pi^* \xi_{\omega_\pi}$, which supports the above assertion.

Since $\gamma \neq 0$, there exist $\psi \in Q_H(M)_n$ and $h_0 \in H$ such that $(\gamma(\psi)h_0, h_0) \neq 0$. Let $\psi = V^* \pi V$ be the canonical representation of ψ where $\pi : M \to B(K)$ is a normal representation of M on a Hilbert space K and $V \in B(H, K)$. We set $K_0 := [\pi(M)Vh_0] \subset K$, $K_1 := K \ominus K_0$ and $\pi_0 := \pi|_{K_0}, \pi_1 := \pi|_{K_1}$. Then, π_0 is a cyclic representation on K_0 with cyclic vector $\xi_0 := Vh_0$. By Prop. 1.4.B, ψ can be written as a CP-convex combination

$$\psi = \sum_{\alpha \in \{\alpha_0\} \cup \Lambda} (W_\alpha^* V)^* (W_\alpha^* \pi_\alpha W_\alpha)(W_\alpha^* V)$$

where $\pi_0 = \pi_{\alpha_0}$, $\pi_1 = \oplus_{\alpha \in \Lambda} \pi_\alpha$, $W_\alpha^* \pi_\alpha W_\alpha \in Rep_c(M : H)_n$ for $\alpha \in \{\alpha_0\} \cup \Lambda$, and W_α is a partial isometry from H onto H_{π_α}. Then,

$$\begin{aligned} 0 \neq (\gamma(\psi)h_0, h_0) &= ((W_{\alpha_0}^* V)^* \gamma(W_{\alpha_0}^* \pi_0 W_{\alpha_0})(W_{\alpha_0}^* V)h_0, h_0) \\ &\quad + \sum_{\alpha \in \Lambda} ((W_\alpha^* V)^* \gamma(W_\alpha^* \pi_\alpha W_\alpha)(W_\alpha^* V)h_0, h_0) \end{aligned}$$

Since $W_\alpha^* V h_0 = 0$ for $\alpha \in \Lambda$ (noting that $Vh_0 \in K_0 = K_1^\perp$), this reduces to

$$0 \neq (\gamma(W_{\alpha_0}^* \pi_{\alpha_0} W_{\alpha_0}) W_{\alpha_0}^* (Vh_0), W_{\alpha_0}^* (Vh_0)).$$

Noting that $W_{\alpha_0}^* \pi_0 W_{\alpha_0} \in Rep_c(M:H)_n$ with cyclic vector $W_{\alpha_0}^*(Vh_0)$, we conclude that the right hand side of this expression must be zero, which is a contradiction. Hence j is injective.

Since the diagram is commutative, k is bijective and j is injective, we conclude that i is surjective. Since we already showed that i is injective, we have proved that i is bijective.

As the second step, we show that $AB(Q_H(M)_n, B(H))$ is a W*-algebra.

We first show that the definition of the product is well defined, i.e., it does not depend on any particular CP-decomposition of ψ. Assume that π has two different cyclic decompositions, i.e., $\pi = \oplus_{\alpha\in\Lambda_1}\pi_\alpha = \oplus_{\beta\in\Lambda_2}\pi_\beta$. According to this, we have two different ways of CP-decomposition, i.e.,

$$\psi = \sum_{\alpha\in\Lambda_1} V_\alpha^* \tilde{\pi}_\alpha V_\alpha = \sum_{\beta\in\Lambda_2} V_\beta^* \tilde{\pi}_\beta V_\beta$$

$$\begin{aligned} \text{with}\qquad & \tilde{\pi}_\alpha = W_\alpha^* \pi_\alpha W_\alpha \in Rep_c(M:H)_n, \qquad V_\alpha = W_\alpha^* V \in B(H) \\ & \tilde{\pi}_\beta = W_\beta^* \pi_\beta W_\beta \in Rep_c(M:H)_n, \qquad V_\beta = W_\beta^* V \in B(H), \end{aligned}$$

where $W_\alpha : H \to H_{\pi_\alpha}$ [resp. $W_\beta : H \to H_{\pi_\beta}$] is a partial isometry from H onto H_{π_α} [resp. H_{π_β}] (cf. Proof of Prop. 1.4). From the first step, for γ_1 and γ_2, there correspond elements a_1 and a_2 in M, so that we have

$$\begin{aligned} \sum_{\alpha\in\Lambda_1} V_\alpha^* \gamma_1(\tilde{\pi}_\alpha)\gamma_2(\tilde{\pi}_\alpha) V_\alpha &= \sum_{\alpha\in\Lambda_1} V_\alpha^* \tilde{\pi}_\alpha(a_1)\tilde{\pi}_\alpha(a_2) V_\alpha \\ &= \sum_{\alpha\in\Lambda_1} V_\alpha^* \tilde{\pi}_\alpha(a_1 a_2) V_\alpha = \psi(a_1 a_2) = \sum_{\beta\in\Lambda_2} V_\beta^* \tilde{\pi}_\beta(a_1 a_2) V_\beta \\ &= \sum_{\beta\in\Lambda_2} V_\beta^* \tilde{\pi}_\beta(a_1)\tilde{\pi}_\beta(a_2) V_\beta = \sum_{\beta\in\Lambda_2} V_\beta^* \gamma_1(\tilde{\pi}_\beta)\gamma_2(\tilde{\pi}_\beta) V_\beta, \end{aligned}$$

which proves the well defindness of $\gamma_1 \cdot \gamma_2$.

It follows directly from the definition of the product that $\gamma_1 \cdot \gamma_2$ is a bounded CP-affine function, so that $AB(Q_H(M)_n, B(H))$ is a *-algebra, where we define the *-operation by $\gamma^*(\psi) := \gamma(\psi)^*$ for $\psi \in Q_H(M)_n$. It can be easily seen that the norm of CP-affine functions, which was defined in Definition 2.1, is actually a C*-norm. In fact, let $\gamma \in AB(Q_H(M)_n, B(H))$ corresponds to $a_\gamma \in M$. Then, $\|a_\gamma\| = \|\gamma\|_c \leq \|\gamma\| \leq \|a_\gamma\|$, i.e., $\|\gamma\| = \|a_\gamma\|$, where $\|\gamma\|_c := \sup\{\|\gamma(\pi)\|; \pi \in Rep_c(M : H)_n\}$ for $\gamma \in AB(Q_H(M)_n, B(H))$, from which $\|\gamma^*\gamma\| = \|\gamma\|^2$ follows. Since the isomorphism $i : M \to AB(Q_H(M)_n, B(H))$ is an order isomorphism with the natural order, $AB(Q_H(M)_n, B(H))$ is monotone closed, hence a W*-algebra.

The last step is to show that i is a *-isomorphism. Let $\gamma_1 = i(a_1)$ and $\gamma_2 = i(a_2)$ with $a_1, a_2 \in M$, and $\psi \in Q_H(M)_n$ with CP-decomposition

$$\psi = \sum_\alpha V_\alpha \pi_\alpha V_\alpha \quad \text{with } \pi_\alpha \in Rep_c(M : H)_n \text{ and } V_\alpha \in B(H)$$

$$\text{such that } \sum_\alpha V_\alpha^* V_\alpha \leq I_H.$$

Then, by definition,

$$\begin{aligned}(\gamma_1 \cdot \gamma_2)(\psi) &= \sum_\alpha V_\alpha^* \gamma_1(\pi_\alpha)\gamma_2(\pi_\alpha) V_\alpha = \sum_\alpha V_\alpha^* \pi_\alpha(a_1)\pi_\alpha(a_2) V_\alpha \\ &= \sum_\alpha V_\alpha^* \pi_\alpha(a_1 a_2) V_\alpha = \psi(a_1 a_2),\end{aligned}$$

which implies $i(a_1) \cdot i(a_2) = i(a_1 a_2)$. Since $\gamma^* = i(a^*)$ $(a \in A)$ and the linearity is obvious, i is a *-isomorphism. ▮

Remark. The CP-duality theorems (Theorem 2.2.A and B) can be considered as natural extensions of Takesaki's and Bichteler's duality theorems for C*-algebras and their enveloping W*-algebras ([17],[73]), i.e.,

$$A \cong A_w^F(Rep(A:H), B(H)) \qquad (\text{*-isomorphism})$$

$$A^{**} \cong A^F(Rep(A:H), B(H)) \qquad (\text{*-isomorphism})$$

where $A^F(Rep(A:H), B(H))$ denotes the set of all functions $\gamma : Rep(A:H) \to B(H)$ which satisfy

(i) $\gamma(\pi) = p_\pi \gamma(\pi) p_\pi$ where p_π is the projection of H onto H_π

(ii) $\gamma(u^*\pi u) = u^*\gamma(\pi)u$ for all unitary u on H

(iii) $\gamma(\pi_1 \oplus \pi_2) = \gamma(\pi_1) \oplus \gamma(\pi_2)$ if $\pi_1 \oplus \pi_2 \in Rep(A:H)$

(iv) $\sup\{\|\gamma(\pi)\|; \pi \in Rep(A:H)\} < \infty$

and $A_w^F(Rep(A:H), B(H))$ denotes the BW-w continuous elements in $A^F(Rep(A:H), B(H))$. We note that Theorem 2.2.A can be proved directly from the above Takesaki duality theorem for C*-algebras by considering the following diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[dr, "t"'] & AC(Q_H(A), B(H)) \arrow[d, "r"] \\
 & A_w^F(Rep(A:H), B(H))
\end{tikzcd}$$

where t denotes the Takesaki's duality, and r is the restriction map. However, our proof of Theorems 2.2.A-B simplifies the arguments and allows us to state the duality for general W*-algebras. It should be noted that the CP-duality, which interpolates Kadison's duality and Takesaki's duality (cf. above diagrams), provides us with both the convexity theoretical aspects and the algebraic aspects of C*-algebras.

We next discuss the dual maps between CP-state spaces induced from *-isomorphisms. We begin with the following definition.

DEFINITION 2.3. *Let A and B be C^*-algebras and let H be a Hilbert space, and let $C_H(A)$ and $C_H(B)$ be CP-convex subsets of $CP(A : B(H))$ and $CP(B : B(H))$ respectively. A map $\Phi : C_H(A) \to C_H(B)$ is defined to be* **CP-affine**, *if*

$$\psi = \sum_\alpha S_\alpha^* \psi_\alpha S_\alpha \quad \textit{with } \psi_\alpha \in C_H(A) \textit{ and } S_\alpha \in B(H) \textit{ such that } \sum_\alpha S_\alpha^* S_\alpha \leq I_H,$$

then we have

$$\Phi(\psi) = \sum_\alpha S_\alpha^* \Phi(\psi_\alpha) S_\alpha.$$

In scalar convexity theory, every affine w*-homeomorphism between quasi-state spaces induces a Jordan isomorphism of the algebras ([51]). In our setting of CP-convexity, we can show the following improvement.

THEOREM 2.4.A. *Let A and B be C^*-algebras, and H be a Hilbert space with $dim H \geq \sup\{\alpha_c(A), \alpha_c(B)\}$. Then, $Q_H(A)$ and $Q_H(B)$ are CP-affine BW-homeomorphic if and only if A and B are *-isomorphic.*

THEOREM 2.4.B. *Let M and N be W^*-algebras, and H be a Hilbert space with $dim H \geq \sup\{\alpha_c(M)_n, \alpha_c(N)_n\}$. Then, $Q_H(M)_n$ and $Q_H(N)_n$ are CP-affine isomorphic if and only if M and N are *-isomorphic.*

Schematically, Kadison's theorem establishes the correspondence :

Jordan structure of the algebras $\longleftrightarrow$ scalar convexity in the state spaces while our theorems imply the refined correlation :

C*-structure of the algebras $\longleftrightarrow$ CP-convexity in the CP-state spaces

In order to prove the above theorems, we need the following propositions.

PROPOSITION 2.5.A. *Let A and B be C*-algebras, and H be a Hilbert space with $dimH \geq \sup\{\alpha_c(A), \alpha_c(B)\}$. Then every CP-affine isomorphism $\Theta : Q_H(A) \to Q_H(B)$ maps Rep$(A : H)$ onto Rep$(B : H)$ bijectively.*

PROPOSITION 2.5.B. *Let M and N be W*-algebras, and H be a Hilbert space with $dimH \geq \sup\{\alpha_c(M)_n, \alpha_c(N)_n\}$. Then every CP-affine isomorphism $\Theta : Q_H(M)_n \to Q_H(N)_n$ maps Rep$(M : H)_n$ onto Rep$(N : H)_n$ bijectively.*

PROOF: We first prove Part A. Let $\pi \in Rep(A : H)$, and let $\Theta(\pi) = V^*\rho V \in Q_H(B)$ where $\rho : B \to B(H_\rho)$ is a representation and $V \in B(H, H_\rho)$. We first show that, if $dimH_\rho \leq dimH$, then $\Theta(\pi) \in Rep(B : H)$ with $H_{\Theta(\pi)} = H_\pi$.

Let $V = v|V|$ be the polar decomposition of V. From the above assumption on the dimension of H_ρ, there exists a partial isometry $\tilde{v} : H \to H_\rho$ of H onto H_ρ which extends v, i.e., $\tilde{v}|_{im|V|} = v$. Then,

$$\Theta(\pi) = |V|v^*\rho v|V| = |V|\tilde{v}^*\rho\tilde{v}|V|. \tag{1}$$

We will show $|V| = p_\pi$, where p_π is the projection of H onto H_π, from which $\Theta(\pi) \in Rep(B : H)$ and $H_{\Theta(\pi)} = H_\pi$ follow, since $\tilde{v} : H \to H_\rho$ is a partial isometry of H onto H_ρ.

We first observe that $\pi = p_\pi \pi p_\pi$. Hence, since Θ is CP-affine,

$$\Theta(\pi) = p_\pi\Theta(\pi)p_\pi = p_\pi|V|\tilde{v}^*\rho\tilde{v}|V|p_\pi. \tag{2}$$

Combining (1) and (2), we conclude that $|V| \leq p_\pi$, since $ess.dom(|V|) \subset H_\pi$ and $\||V|\| = \|V\| \leq 1$.

Let $\Theta^{-1}(\tilde{v}^*\rho\tilde{v}) = W^*\tau W \in Q_H(A)$ where $\tau : A \to B(H_\tau)$ is a representation and $W \in B(H, H_\tau)$. Since Θ is an CP-affine isomorphism,

$$\pi = \Theta^{-1}(\Theta\pi) = \Theta^{-1}(|V|\tilde{v}^*\rho\tilde{v}|V|) = |V|\Theta^{-1}(\tilde{v}^*\rho\tilde{v})|V| = |V|W^*\tau W|V|,$$

so we can observe that, if A has a unit e,

$$p_\pi = \pi(e) = |V|W^*\tau(e)W|V| = |V|W^*W|V| \leq |V|^2 \leq p_\pi,$$

from which $|V| = p_\pi$ follows. If A is not unital, we can take an approximate unit (e_α) to deduce, as above,

$$\pi(e_\alpha) = |V|W^*\tau(e_\alpha)W|V| \leq |V|W^*W|V| \leq |V|^2 \leq p_\pi,$$

so that, in the limit,

$$p_\pi = \sup_\alpha \pi(e_\alpha) \leq |V|^2 \leq p_\pi,$$

which proves $|V| = p_\pi$.

We next consider the case where $dimH_\rho > dimH$. Let $\rho = \oplus_\alpha \rho_\alpha$ be a cyclic decomposition of ρ and let p_α be the projection of H_ρ onto H_{ρ_α}. Since $dimH > \alpha_c(B)$, there exists a partial isometry $v_\alpha : H \to H_{\rho_\alpha}$ of H onto H_{ρ_α}, i.e.. $v_\alpha v_\alpha^* = p_\alpha$. Then,

$$\Theta(\pi) = V^*\rho V = \sum_\alpha V^*p_\alpha\rho_\alpha p_\alpha V = \sum_\alpha V^*v_\alpha(v_\alpha^*\rho_\alpha v_\alpha)v_\alpha^*V,$$

where $v_\alpha^* \rho_\alpha v_\alpha \in Rep_c(B : H)$. Let $\Theta^{-1}(v_\alpha^* \rho_\alpha v_\alpha) = W_\alpha^* \tau_\alpha W_\alpha$ and observe that

$$\begin{aligned} \pi = \Theta^{-1}(\Theta(\pi)) &= \sum_\alpha (v_\alpha^* V)^* \Theta^{-1}(v_\alpha^* \rho_\alpha v_\alpha)(v_\alpha^* V) \\ &= \sum_\alpha (v_\alpha^* V)^* W_\alpha^* \tau_\alpha W_\alpha (v_\alpha^* V). \end{aligned} \tag{3}$$

We note that these expressions in the above sum are minimal. In fact, if it is not minimal, we could replace τ_α by $\tau'_\alpha \subset \tau_\alpha$; however

$$\begin{aligned} \Theta(\pi) &= \sum_\alpha (v_\alpha^* V)^* \Theta(W_\alpha^* \tau'_\alpha W_\alpha)(v_\alpha^* V) \\ &\leq \sum_\alpha (v_\alpha^* V)^* \Theta(W_\alpha^* \tau_\alpha W_\alpha)(v_\alpha^* V) \\ &= \sum_\alpha (v_\alpha^* V)^* (v_\alpha^* \rho_\alpha v_\alpha)(v_\alpha^* V) = \Theta(\pi), \end{aligned}$$

where the expressions in the last sum are minimal, so that we conclude $\tau'_\alpha = \tau_\alpha$, which implies that the expressions in (3) are minimal.

Now that $(v_\alpha V)^* W_\alpha^* \tau_\alpha W_\alpha (v_\alpha^* V) \leq \pi$, from [11, Theorem 1.4.2], there exists $T_\alpha \in (\pi(A)')_+$, such that

$$(v_\alpha^* V)^* W_\alpha \tau_\alpha W_\alpha (v_\alpha^* V) = T_\alpha^{\frac{1}{2}} \pi T_\alpha^{\frac{1}{2}}.$$

Since the expression in the left is minimal, there exists a partial isometry $U_\alpha : H_{\tau_\alpha} \to H_\pi$ such that

$$U_\alpha \tau_\alpha U_\alpha^* = \pi|_{[\pi(A) T_\alpha^{\frac{1}{2}} H]} \quad \text{and} \quad W_\alpha v_\alpha^* V = U_\alpha^* T_\alpha^{\frac{1}{2}},$$

hence $dim\, \tau_\alpha \leq dim\, \pi \leq dim H$. Then, from the first part of the present proof, we conclude

$$\Theta^{-1}(v_\alpha^* \rho_\alpha v_\alpha) = W_\alpha^* \tau_\alpha W_\alpha \in Rep(A : H) \quad \text{and} \quad H_{W_\alpha^* \tau_\alpha W_\alpha} = H_{v_\alpha^* \rho_\alpha v_\alpha}.$$

If A is unital, $W_\alpha^* \tau_\alpha(e) W_\alpha = v_\alpha^* v_\alpha$, hence,

$$\pi(e) = \sum_\alpha (v_\alpha^* V)^* W_\alpha^* \tau_\alpha(e) W_\alpha (v_\alpha^* V) = \sum_\alpha (v_\alpha^* V)^* (v_\alpha^* v_\alpha)(v_\alpha^* V) = p_\pi,$$

so that

$$V^* V = V^* \rho(e) V = \sum_\alpha (v_\alpha^* V)^* (v_\alpha^* \rho_\alpha(e) v_\alpha)(v_\alpha^* V) = \sum_\alpha (v_\alpha^* V)^* (v_\alpha^* v_\alpha)(v_\alpha^* V) = p_\pi,$$

i.e., V is a partial isometry. Therefore, we conclude that $\Theta(\pi) = V^* \rho V \in Rep(B : H)$. If A is not unital, we can deduce the same conclusion using the approximate unit (e_α).

Now, we have proved

$$\Theta(Rep(A : H)) \subset Rep(B : H) \quad \text{and} \quad \Theta^{-1}(Rep(B : H)) \subset Rep(A : H),$$

so that, since Θ is one to one, Θ maps $Rep(A : H)$ onto $Rep(B : H)$ bijectively.

Part B is proved similarly. ∎

PROOF OF THEOREM 2.4: We prove Part A. It is straightforward to see that, if A and B are *-isomorphic, then $Q_H(A)$ and $Q_H(B)$ are CP-affine isomorphic. In fact, let $\theta : A \to B$ be a *-isomorphism. We define $\theta^\natural : Q_H(B) \to Q_H(A)$ by $\theta^\natural(\psi) = \psi \circ \theta$ for $\psi \in Q_H(B)$. Then, $\theta^\natural$ is a CP-affine and BW-continuous isomorphim of $Q_H(B)$ to $Q_H(A)$.

Conversely, let $\Theta : Q_H(A) \to Q_H(B)$ be a CP-affine BW-continuous isomorphism. By Theorem 2.2.A, we can identify A with $AC(Q_H(A), B(H))$ and B with $AC(Q_H(B), B(H))$. We define $\Theta^\natural : AC(Q_H(B), B(H)) \to AC(Q_H(A), B(H))$ by $\Theta^\natural(\gamma) = \gamma \circ \Theta$ for $\gamma \in AC(Q_H(B), B(H))$. We have to prove that $\Theta^\natural$ is a *-isomorphism.

We only need to check that $\Theta^\natural$ preserves the C*-product. Indeed let $\psi \in Q_H(A)$ and assume

$$\psi = \sum_\alpha V_\alpha^* \pi_\alpha V_\alpha \quad \text{with } \pi_\alpha \in Rep_c(A : H) \text{ and } V_\alpha \in B(H)$$

$$\text{such that } \sum_\alpha V_\alpha^* V_\alpha \leq I_H.$$

Then, for $\gamma_1, \gamma_2 \in AC(Q_H(B), B(H))$, we obtain upon using the fact that $\Theta(\pi_\alpha) \in Rep(B : H)$ by Proposition 2.5.A,

$$\begin{aligned}
\Theta^\natural(\gamma_1 \cdot \gamma_2)(\psi) &= (\gamma_1 \cdot \gamma_2)(\Theta(\psi)) \\
&= (\gamma_1 \cdot \gamma_2)(\sum_\alpha V_\alpha^* \Theta(\pi_\alpha) V_\alpha) \\
&= \sum_\alpha V_\alpha^* \gamma_1(\Theta(\pi_\alpha)) \cdot \gamma_2(\Theta(\pi_\alpha)) V_\alpha \\
&= \sum V_\alpha^* \ \Theta^\natural(\gamma_1)(\pi_\alpha) \cdot \Theta^\natural(\gamma_2)(\pi_\alpha) V_\alpha \\
&= (\Theta^\natural(\gamma_1) \cdot \Theta^\natural(\gamma_2))(\sum_\alpha V_\alpha^* \pi_\alpha V_\alpha) \\
&= (\Theta^\natural(\gamma_1) \cdot \Theta^\natural(\gamma_2))(\psi).
\end{aligned}$$

This completes the proof of Part A.

Part B is proved by the similar arguments using Proposition 2.5.B. ▮

Remark. A preliminary report on parts of Chapters 1 and 2 is to appear in [37].

CHAPTER 3

CP-DUALITY FOR JC- AND JW-ALGEBRAS

In this chapter, we generalize the notion of completely positive maps to the category of JC-algebras, and discuss CP-convexity and CP-duality for JC- and JW-algebras. These results will be essentially used for our formulation of orientability conditions in the CP-geometrical setting in Chapter 7.

We recall that a normed Jordan algebra is called a *JC-algebra* [resp. *JW-algebra*] if it is isometrically isomorphic to a norm closed [resp. weakly closed] Jordan subalgebra of self-adjoint bounded linear operators on a (complex) Hilbert space. JW-algebras are also characterized as the JC-algebras with predual. For each JC-algebra A, there exist a C*-algebra $C^*(A)$, called the *enveloping C*-algebra* of A, and an injective Jordan homomorphism $\phi_A : A \to C^*(A)$ such that $\phi_A(A)$ generates $C^*(A)$ as a C*-algebra, and any Jordan homomorphism $\rho : A \to \mathcal{A}_{sa}$, where $\mathcal{A}$ is a C*-algebra, extends uniquely to a *-homomorphism $\bar{\rho} : C^*(A) \to \mathcal{A}$ (cf. [4; Th. 5.1], [45; Th. 7.1.8]). Similarly, for each JW-algebra M, there exist the *enveloping W*-algebra* $W^*(M)$ and an injective normal homomorphism $\phi_M : M \to W^*(M)$ such that $\phi_M(M)$ generates $W^*(M)$, and any normal Jordan homomorphism $\rho : M \to \mathcal{M}_{sa}$, where $\mathcal{M}$ is a W*-algebra, extends uniquely to a normal *-homomorphism $\bar{\rho} : W^*(M) \to \mathcal{M}$ (cf. [45; Th. 7.1.8-9]). For the enveloping C*-algebra $C^*(A)$ [resp. enveloping W*-algebra $W^*(M)$], there exists a unique *-anti-automorphism Φ of $C^*(A)$ [resp. $W^*(M)$] of order 2 which leaves $\phi_A(A)$ [resp. $\phi_M(M)$] pointwise invariant ([4; Cor. 5.2], [45; Th. 7.1.9]). Following [44], the *universal tensor product* $A\tilde{\otimes}B$ of JC-algebras A and B is defined to be the JC-subalgebra of $C^*(A) \otimes_{max} C^*(B)$ generated by the subspace $\phi_A(A) \otimes \phi_B(B)$.

We then define the completely positive maps for JC-algebras as follows.

DEFINITION 3.1. *A positive real linear map ψ between two JC-algebras A and B is defined to be* **completely positive**, *if, for every integer $n \geq 1$, the corresponding linear map*

$$\psi_n : [a_{ij}] \in A \otimes M_n(\mathbb{C})_{sa} \to [\psi(a_{ij})] \in B \otimes M_n(\mathbb{C})_{sa}$$

extends to a positive linear map

$$\tilde{\psi}_n : A\tilde{\otimes}M_n(\mathbb{C})_{sa} \to B\tilde{\otimes}M_n(\mathbb{C})_{sa}.$$

We denote by $CP(A, B(H)_{sa})$ the set of all completely positive linear maps from a JC-algebra A into the JC-algebra $B(H)_{sa}$ of all self-adjoint bounded linear operators on a Hilbert space H, and by $Q_H(A)$ the **CP-state space** *of A for H, i.e.,*

$$Q_H(A) = \{\psi \in CP(A, B(H)_{sa}); \|\psi\| \leq 1\}.$$

Similarly, we define, for a JW-algebra M, $CP(M, B(H)_{sa})_n$, $Q_H(M)_n$ as the normal parts.

We first generalize the Stinespring representation theorem as follows.

THEOREM 3.2.A. *Let A be a JC-algebra and H be a Hilbert space. Then, $\psi \in CP(A, B(H)_{sa})$ if and only if ψ can be written in the form*

$$\psi(a) = V^* \rho(a) V \qquad \text{for all } \; a \in A$$

where ρ is a Jordan representation of A on some Hilbert space K and $V \in B(H, K)$.

THEOREM 3.2.B. *Let M be a JW-algebra and H be a Hilbert space. Then, $\psi \in CP(M, B(H)_{sa})_n$ if and only if ψ can be written in the form*

$$\psi(a) = V^* \rho(a) V \qquad \text{for all } \; a \in M$$

where ρ is a normal Jordan representation of M on some Hilbert space K and $V \in B(H, K)$.

PROOF: We prove Part A. Let $\psi \in CP(A, B(H)_{sa})$ and consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\phi_A"] \arrow[d, "\psi"'] & \phi_A(A) \arrow[r, "\subset"] \arrow[d, "\varphi"] & C^*(A) \arrow[d, "\Psi"] \\
B(H)_{sa} \arrow[r, "\phi_B"'] & \phi_B(B(H)_{sa}) \arrow[r, "\subset"'] & C^*(B(H)_{sa})
\end{tikzcd}$$

where ϕ_A and ϕ_B denote the canonical injections of A and $B(H)_{sa}$ respectively, and φ is defined by $\varphi = \phi_B \circ \psi \circ \phi_A^{-1}$. It is obvious that φ is positive since ϕ_A and ϕ_B are Jordan isomorphisms. We shall show that φ can be extended canonically to a completely positive map $\Psi : C^*(A) \to C^*(B(H)_{sa})$ in the C*-algebra sense. We use the following result proved in [44; Lemma 5.4]; if B is a JC-algebra and $\mathcal{B}$

is a C*-algebra with no scalar representation, then $B\tilde{\otimes}\mathcal{B}_{sa} \cong (C^*(B) \otimes_{max} \mathcal{B})_{sa}$. We substitute here $B = A$ and $\mathcal{B} = M_n(\mathbf{C})$ $(n \geq 2)$, and then $B = B(H)_{sa}$ and $\mathcal{B} = M_n(\mathbf{C})$ $(n \geq 2)$, and get the following diagram:

$$\begin{tikzcd}
A\tilde{\otimes}M_n(\mathbf{C})_{sa} \arrow[r, "\cong"] \arrow[d, "\tilde{\psi}_n"'] & (C^*(A) \otimes M_n(\mathbf{C}))_{sa} \arrow[d, "\Psi_n"] & \phi_A(A) \otimes M_n(\mathbf{C})_{sa} \arrow[l, "\supset"'] \arrow[d, "\varphi_n"] \\
B(H)_{sa}\tilde{\otimes}M_n(\mathbf{C})_{sa} \arrow[r, "\cong"'] & (C^*(B(H)_{sa}) \otimes M_n(\mathbf{C}))_{sa} & \phi_B(B(H)_{sa}) \otimes M_n(\mathbf{C})_{sa} \arrow[l, "\supset"]
\end{tikzcd}$$

where $\tilde{\psi}_n$ denotes the positive linear map defined in Definition 3.1, Ψ_n denotes the maps defined for $\tilde{\psi}_n$ by the bijection maps $\cong$ of the diagram, and φ_n denotes the induced map corresponding to φ. (We omitted the sufix max under the tensor product $\otimes$ since $M_n(\mathbf{C})$ is a nuclear C*-algebra, for which the tensor product is unique.) From the above diagrams, it follows that φ is completely positive since ψ is completely positive, hence so is the complex extension $\tilde{\varphi} : \phi_A(A) + i\phi_A(A) \to \phi_B(B(H)_{sa}) + i\phi_B(B(H)_{sa})$.

If A is unital (with identity e), then $\phi_A(e)$ is the identity of $C^*(A)$ since $C^*(A)$ is generated by $\phi_A(A)$, so that $\phi_A(A)^{\mathbf{C}} := \phi_A(A) + i\phi_A(A)$ is an operator system of $C^*(A)$, i.e., $\phi_A(A)^{\mathbf{C}}$ is a selfadjoint subspace of $C^*(A)$ containing the identity of $C^*(A)$. We also note that $C^*(B(H)_{sa}) = B(H) \oplus B(H)^\circ$ (where $B(H)^\circ$ denotes the opposite C*-algebra of $B(H)$), which is a direct sum of injective C*-algebras, so that it is injective. Hence, the completely positive map $\tilde{\varphi}$ from the operator system $\phi_A(A)^{\mathbf{C}}$ to $C^*(B(H)_{sa})$ has a completely positive extension $\Psi : C^*(A) \to C^*(B(H)_{sa})$. (This is essentially due to Arveson's extension theorem. cf. [11; Th.

1.2.7] or [59; Th. 6.5] for the general version.)

If A is not unital, we take the JC-algebra A_e obtained by adjoining an identity e to A, i.e., $A_e = A + \mathbf{R}e$. Let $(u_i)_{i \in I}$ be an approximate identity of A (e.g., [45:3.5.4]), and define $\psi_e(a + \alpha e) := \psi(a) + \alpha(w - \lim_i \psi(u_i))$ for $\alpha \in \mathbf{R}$, so that $\varphi_e(c + \alpha \phi_{A_e}(e)) = \varphi(c) + \alpha(w - \lim_i \varphi(\phi_A(u_i))$ for $c \in \phi_A(A)$ and $\alpha \in \mathbf{R}$. Then, $\bar{\varphi}_e : \phi_{A_e}(A_e)^{\mathbf{C}} \to \phi_B(B(H)_{sa})^{\mathbf{C}} \subset C^*(B(H)_{sa})$ is completely positive, hence is extended to a completely positive map $\Psi_e : C^*(A_e) \to C^*(B(H)_{sa})$ as we showed in the unital case. Thus, $\varphi = \varphi_e|_{\phi_A(A)}$ has a completely positive extension $\Psi = \Psi_e|_{C^*(A)} : C^*(A) \to C^*(B(H)_{sa})$.

We have now, from the first of the above diagrams,

$$\Psi(\phi_A(a)) = \phi_B(\psi(a)) \qquad \text{for all} \quad a \in A,$$

and we recall that ϕ_B is given by $\phi_B(b) = b \oplus b^\circ$ for $b \in B(H)_{sa}$ (cf. [45; Th. 7.4.7]), hence

$$\Psi(\phi_A(a)) = \psi(a) \oplus \psi(a)^\circ \qquad \text{for all } a \in A.$$

Since $\Psi \in CP(C^*(A), B(H) \oplus B(H)^\circ)$, we can set

$$\Psi(c) = \bar{\psi}(c) \oplus \bar{\psi}(c)^\circ \qquad \text{for } c \in C^*(A)$$

$$\text{with} \quad \bar{\psi} \in CP(C^*(A), B(H)) \quad \text{such that} \quad \psi = \bar{\psi} \circ \phi_A.$$

Let $\bar{\psi} = V^* \kappa V$ where κ is a *-representation of $C^*(A)$ on a Hilbert space K, and

$V \in B(H,K)$. Then, for any $a \in A$,

$$\begin{aligned}\psi(a) &= (\phi_B^{-1} \circ \Psi \circ \phi_A)(a) = \phi_B^{-1}[\bar{\psi}(\phi_A(a)) \oplus \bar{\psi}(\phi_A(a))^\circ] \\ &= \phi_B^{-1}[V^*(\kappa \circ \phi_A)(a)\, V \oplus (V^*(\kappa \circ \phi_A)(a)\, V)^\circ] \\ &= V^*(\kappa \circ \phi_A)(a)\, V = V^*\rho(a)\, V,\end{aligned}$$

where we set $\rho = \kappa \circ \phi_A$, which is a Jordan representation of A on K.

Conversely, assume that $\psi : A \to B(H)_{sa}$ is written in the form $\psi = V^*\rho V$ where $\rho : A \to B(K)_{sa}$ is a Jordan representation of A on a Hilbert space K and $V \in B(H,K)$. Since ρ is uniquely extended to a *-representation $\kappa : C^*(A) \to B(K)$, ψ extends to a CP-map $\bar{\psi} = V^*\kappa V : C^*(A) \to B(H)$. Define

$$\Psi(c) = \bar{\psi}(c) \oplus \bar{\psi}(c)^\circ \qquad \text{for all } \ c \in C^*(A),$$

then $\Psi \in CP(C^*(A), C^*(B(H)_{sa}))$, so that, from the diagrams, $\psi = \phi_B^{-1} \circ \Psi \circ \phi_A$ and the maps $\tilde{\psi}_n$ induced by Ψ_n are all positive. Hence $\psi \in CP(A, B(H)_{sa})$. This proves Theorem 3.2.A.

Upon using the normal part in the arguments of Theorem 3.2.A, we immediately obtain Theorem 3.2.B. ∎

Remarks. 1. Let $\psi = V^*\rho V \in CP(A, B(H)_{sa})$ where $\rho : A \to B(K)_{sa}$ is a Jordan representation and $V \in B(H,K)$. We denote by $\bar{\rho}$ the canonical extension of ρ, i.e., the unique extension of ρ as a *-representation to $C^*(A)$, and define $\bar{\psi} = V^*\bar{\rho}V \in CP(C^*(A), B(H))$, which we call the *canonical extension* of ψ. Then, we can assume the minimal conditions $H_\rho = [\rho(A)VH]$ and $H_{\bar{\rho}} = [\bar{\rho}(C^*(A))VH]$, and that $H_\rho = H_{\bar{\rho}}$ since $\bar{\rho}$ is an extension of ρ on the same Hilbert space H_ρ.

2. Let $V^*\rho V \in CP(A, B(H)_{sa})$ and let $J : H_\rho \to H_\rho$ be an anti-unitary operator, and define a transposed representation of ρ by $\rho^t = J\rho J$. Then, $V^*\rho^t V \in CP(A, B(H)_{sa})$ since ρ^t is a Jordan representation of A on H_ρ. This makes the situation different from the C*-algebra case where the transposed map was not completely positive.

3. Let $\psi = V^*\rho V \in CP(A, B(H)_{sa})$, then $\|\psi\| = \|V\|^2$. This is proved similarly as in the C*-algebra case due to the existence of approximate identity in JC-algebras.

We now consider CP-convex combinations in the cone $CP(A, B(H)_{sa})$.

PROPOSITION 3.3. *Let A be a JC-algebra and H be a Hilbert space, and let $(\psi_\alpha)_{\alpha\in\Lambda}$ be a bounded family in $CP(A, B(H)_{sa})$. Then, any CP-convex combination*

$$\sum_{\alpha\in\Lambda} S_\alpha^*\psi_\alpha S_\alpha \quad \text{with } S_\alpha \in B(H) \quad \text{such that} \qquad \sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha \le I_H$$

converges in the BW-topology to an element $\psi \in CP(A, B(H)_{sa})$ with $\|\psi\| \le \sup_{\alpha\in\Lambda} \|\psi_\alpha\|$ $(:= r)$.

PROOF: By Theorem 3.2, we can write $\psi_\alpha = V_\alpha^*\rho_\alpha V_\alpha$ where $\rho_\alpha : A \to B(K_\alpha)$ is a Jordan representation and $V_\alpha \in B(H, K_\alpha)$. Let $\bar{\psi}_\alpha = V_\alpha^*\bar{\rho}_\alpha V_\alpha$ be the canonical extension of ψ_α, then $\|\bar{\psi}_\alpha\| = \|V_\alpha\|^2 = \|\psi_\alpha\| \le r$ (cf. Remark 3 above). By Proposition 1.2, $\sum_\alpha S_\alpha^*\bar{\psi}_\alpha S_\alpha$ converges in the BW-topology to an element $\bar{\psi} = V^*\kappa V \in CP(C^*(A), B(H))$ where κ is a *-representation of $C^*(A)$ on some Hilbert space K and $V \in B(H, K)$, and $\|\bar{\psi}\| = \|V\|^2 \le r$. Hence $\sum_\alpha S_\alpha^*\psi_\alpha S_\alpha$ converges in the BW-topology to $\psi := \bar{\psi}|_A = V^*\kappa|_A V = V^*\rho V$ where $\rho = \kappa|_A$ is a Jordan representation of A on K, so $\psi \in CP(A, B(H)_{sa})$ with $\|\psi\| = \|V\|^2 \leq r$. ∎

Let A be a JC-algebra and ρ be a Jordan representation of A on a complex Hilbert space H_ρ. Then the canonical extension $\bar{\rho}$ is a *-representation of $C^*(A)$ on H_ρ, hence has a cyclic decomposition $\bar{\rho} = \oplus_\alpha \bar{\rho}_\alpha$. Let $\rho_\alpha := \bar{\rho}_\alpha|_A$, then ρ_α is a *cyclic* Jordan representation with $H_{\rho_\alpha} = H_{\bar{\rho}_\alpha}$; in fact, from Remark 1 to Theorem 3.2, if ξ_α is a cyclic vector for $\bar{\rho}_\alpha$, $H_{\bar{\rho}_\alpha} = [\bar{\rho}_\alpha(C^*(A)\xi_\alpha] = [\rho_\alpha(A)\xi_\alpha] = H_{\rho_\alpha}$. Hence ρ has a cyclic decomposition $\rho = \oplus_\alpha \rho_\alpha$. From this it is reasonable to define the *cyclic dimension* of A by $\alpha_c(A) := \alpha_c(C^*(A))$. Similarly, for a JW-algebra M, we can define the *normal cyclic demension* by $\alpha_c(M)_n := \alpha_c(W^*(M))_n$.

We note here that, if ρ is a cyclic Jordan representation of a JC-algebra A with a cyclic vector ξ, then $[\rho(A)\xi]_{\mathbf{C}} = H_\rho$; however, $[\rho(A)\xi]_{\mathbf{R}} \neq H_\rho$ in general, where we denote by $[\ ]_{\mathbf{C}}$ [resp. $[\ ]_{\mathbf{R}}$] the closed complex [resp. real] linear hull. In this sense, ξ is not a "real" cyclic vector. For example, let $A = M_2(\mathbf{R})_s$ (the set of all 2×2 symmetric matrices), and $\rho = id$ be the identity representation of A on $\mathbf{R}^2 \subset \mathbf{C}^2$. Take a vector $\xi \in \mathbf{R}^2$ ($\xi \neq 0$), then $[\rho(A)\xi]_{\mathbf{R}} = \mathbf{R}^2$, while $H_\rho = [\rho(A)\xi]_{\mathbf{C}} = \mathbf{C}^2$. Consequently, when we say that ξ is a cyclic vector for the Jordan representation ρ of a JC-algebra A, we mean that $[\rho(A)\xi]_{\mathbf{C}} = H_\rho$.

In the following, we use the notations $JRep(A : H)$ [resp. $JRep_c(A : H)$] to denote the set of all Jordan representations [resp. Jordan cyclic representations] of a JC-algebra A on a Hilbert space H. Similarly, we define $JRep(M : H)_n$, $JRep_c(M : H)_n$ for a JW-algebra M as the normal part of the representations.

PROPOSITION 3.4.A. *Let A be a JC-algebra and H be a Hilbert space with $dim H \geq \alpha_c(A)$. Then,*

$$Q_H(A) = CP - conv\ JRep_c(A : H)$$

PROOF: By Proposition 1.4.A, $Q_H(C^*(A)) = CP - conv\ Rep_c(C^*(A) : H)$. Hence, noting $\alpha_c(A) = \alpha_c(C^*(A))$ by definition, we have $Q_H(A) = r(Q_H(C^*(A)) = CP - conv\ r(Rep_c(C^*(A) : H)) = CP - conv\ JRep_c(A : H)$, since the restriction map is CP-affine. ∎

Similarly, we have

PROPOSITION 3.4.B. *Let M be a JW-algebra and H be a Hilbert space with $dim H \geq \alpha_c(M)_n$. Then,*

$$Q_H(M)_n = CP - conv\ JRep_c(M : H)_n$$

We next show that the GNS-representation for C*-algebras can be generalized to JC-algebras.

PROPOSITION 3.5. *Let A be a JC-algebra, and $\omega \in Q(A)$ be a quasi-state of A. Then, there exists a Jordan representation ρ_ω of A on a Hilbert space H_ω and a cyclic vector $\xi_\omega \in H_\omega$ for ρ_ω with $\|\xi_\omega\|^2 = \|\omega\|$, such that*

$$\omega(a) = (\rho_\omega(a)\xi_\omega, \xi_\omega) \quad \text{for all} \quad a \in A$$

If M is a JW-algebra and $\omega \in Q(M)_n$, then we have the same expression with ρ_ω normal.

PROOF: We regard A as a subspace of $C^*(A)_{sa}$ by the canonical image. Then, we can extend $\omega \in Q(A)$ to $C^*(A)_{sa}$ which we denote by $\bar{\omega} \in Q(C^*(A))$.

(The existence of such $\bar{\omega}$ is proved as in the C*-algebra case.) From the GNS-construction for the C*-algebra $C^*(A)$, there exists a *-representation κ_ω of $C^*(A)$ on a Hilbert space H_ω and a cyclic vector $\xi_\omega \in H_\omega$ for κ_ω such that

$$\bar{\omega}(c) = (\kappa_\omega(c)\xi_\omega, \xi_\omega) \quad \text{for all } \ c \in C^*(A)$$

with $\|\xi_\omega\|^2 = \|\bar{\omega}\|$. Let $\rho_\omega = \kappa|_A$, then ρ_ω is a Jordan representation of A on H_ω, and, since $\bar{\omega}|_A = \omega$, we have

$$\omega(a) = (\rho_\omega(a)\xi_\omega, \xi_\omega) \quad \text{for all } \ a \in A.$$

It follows from Remark 1 to Theorem 3.2 that ξ_ω is a cyclic vector for ρ_ω since κ_ω is the canonical extension of ρ_ω.

If $A = M$ is a JW-algebra and ω is normal, then κ_ω, hence ρ_ω is normal. ∎

Similarly as in the C*-algebra case, quasi-states are characterized as one-dimensional CP-states.

PROPOSITION 3.6. *Let A be a JC-algebra, and H be a Hilbert space. Then,*

$$Q(A) \cong \{\hat{\psi}; \psi = V^*\rho V \in Q_H(A), dim[imV] = 1\} := \hat{Q}_{H,1}(A)$$

where $\hat{}$ represents the equivalence class, and the correspondence is given by

$$\omega \in Q(A) \leftrightarrow \hat{\psi}_\omega := (\omega(\cdot)P)\hat{} \in \hat{Q}_{H,1}(A)$$

where P is a one dimensional projection.

If M is a JW-algebra, we have $Q(M)_n \cong \hat{Q}_{H,1}(M)_n$ with the same correspondence.

Proof: From the inclusion $A \subset C^*(A)_{sa}$, we have $Q(A) = r(Q(C^*(A))$, where r denotes the restriction map. Then the isomorphism $Q(A) \cong \hat{Q}_{H,1}(A)$ follows from Proposition 1.5 and Propositions 3.2.

To see the correspondence, note, from Proposition 3.5, that $\omega \in Q(A)$ is of the form

$$\omega(a) = (\rho_\omega(a)\xi_\omega, \xi_\omega) \quad \text{for all } a \in A$$

where ρ_ω is a Jordan representation of A on a Hilbert space H_ω and ξ_ω is a cyclic vector for ρ_ω. Let $\bar{\rho}_\omega$ be the canonical extension of ρ_ω, and $\bar{\omega}$ be the corresponding extension of ω, then

$$\bar{\omega}(c) = (\bar{\rho}_\omega(c)\xi_\omega, \xi_\omega) \quad \text{for all } c \in C^*(A).$$

In Theorem 1.5, we showed that this expression is equivalent to

$$\bar{\omega}(\cdot)P = V_\omega^* \bar{\rho}_\omega V_\omega \ \ (:= \psi_{\bar{\omega}})$$

where P is a one dimensional projection on H and V_ω is defined by

$$V_\omega(k) = (k, h)\xi_\omega \quad \text{with } h \in imP \ \ (\|h\| = 1) \quad \text{and } k \in H.$$

Then, restricting to A,

$$\omega(\cdot)P = V_\omega^* \rho_\omega V_\omega \ \ (:= \psi_\omega)$$

This shows the correspondence. ∎

Remark. As we mentioned, the canonical extension of a Jordan repesentation of a JC-algebra A to its enveloping C*-algebra $C^*(A)$ as a *-representation is unique by definition. However, it does not imply the uniqueness of any extension of CP-maps. For example, consider $A = M_2(\mathbf{R})_s$, then $C^*(A) = M_2(\mathbf{C})$, and then by Proposition 3.6,

$$CP_1(A, B(H)_{sa}) \cong CP(A, \mathbf{R}) \cong (A^*)^+ \cong (\mathbf{R}^3)^+$$

$$CP_1(C^*(A), B(H)) \cong CP(C^*(A), \mathbf{C}) \cong (C^*(A)^*)^+ \cong (\mathbf{R}^4)^+$$

where CP_1 denotes the one dimensional CP-maps. This shows that the restriction map $r : CP(C^*(A), B(H)) \to CP(A, B(H)_{sa})$ is not one to one.

We shall now establish the CP-duality for JC- and JW-algebras. We employ the same definition as Definition 2.1 to define $B(H)_{sa}$-valued CP-affine functions. The notations $AC(Q_H(A), B(H)_{sa})$ for a JC-algebra A, and $AB(Q_H(M)_n, B(H)_{sa})$ for a JW-algebra M are defined similarly as for C*- and W*-algebras.

PROPOSITION 3.7.A. *Let A be a JC-algebra, and H be a Hilbert space with $dim H \geq \alpha_c(A)$. Then, $AC(Q_H(A), B(H)_{sa})$ is a Jordan algebra with the following product; for $\gamma_1, \gamma_2 \in AC(Q_H(A), B(H)_{sa})$ and $\psi = V^*\rho V \in Q_H(A)$ with CP-decomposition*

$$\psi = \sum_\alpha V_\alpha^* \rho_\alpha V_\alpha \quad \textit{where } \rho_\alpha \in JRep_c(A:H) \textit{ with } \rho \simeq \oplus_\alpha \rho_\alpha$$
$$\textit{and } V_\alpha \in B(H) \textit{ such that } \sum_\alpha V_\alpha^* V_\alpha \leq I_H \ (\textit{cf. Prop. 3.4.A}),$$

we define the Jordan product of γ_1 and γ_2 by

$$(\gamma_1 \circ \gamma_2)(\psi) := \sum_\alpha V_\alpha^* \gamma_1(\rho_\alpha) \circ \gamma_2(\rho_\alpha) V_\alpha.$$

We then have

$$A \cong AC(Q_H(A), B(H)_{sa}) \qquad \text{(Jordan isomorphism)}$$

PROPOSITION 3.7.B. *Let M be a JW-algebra, and H be a Hilbert space with $dimH \geq \alpha_c(M)_n$. Then, $AB(Q_H(M)_n, B(H)_{sa})$ is a JW-algebra with respect to the product defined in Proposition 3.7.A, where $JRep_c(A : H)$ is replaced by $JRep_c(M : H)_n$, and we have*

$$M \cong AB(Q_H(M)_n, B(H)_{sa}) \qquad \text{(Jordan isomorphism).}$$

PROOF: We shall prove Part B, from which Part A follows by restricting our objects to BW-w continuous elements.

We first show that there exists a one to one correspondence between M and $AB(Q_H(M)_n, B(H)_{sa})$. Consider the following diagram:

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "k"'] & AB(Q_H(M)_n, B(H)_{sa}) \arrow[d, "j"] \\ & A_0^b(Q(M)_n, \mathsf{R}) \end{tikzcd}$$

where $A_0^b(Q(M)_n, \mathsf{R})$ denotes the set of all real valued bounded affine functions on $Q(M)_n$, and i assignes an evaluation map, and j is defined for $\gamma \in AB(Q_H(M)_n, B(H)_{sa})$ and $\omega \in Q(M)_n$ by

$$j(\gamma)(\omega) = (\gamma(\psi_\omega)h, h) \quad \text{with } h \in ess.supp(\psi_\omega), \ \|h\| = 1.$$

The isomorphism k is the generalization of Kadison's function representation, which has no limitation to be extended to JW-algebras. (cf. [60;3.10.3] for the proof for

C*-algebras, and [5] for unital JB-algebras). It is shown that the above diagram commutes as in the proof of Theorem 2.2.

It is obvious that i is injective. To prove that i is surjective, we need to show that j is injective. Let $\gamma \in AB(Q_H(M)_n, B(H)_{sa})$ and assume $j(\gamma) = 0$. Then, we can trace the same argument in the proof of W*-algebra case (Theorem 2.2) using Propositions 3.5 and 3.6, and conclude $(\gamma(\rho)\xi, \xi) = 0$ for all pair of normal cyclic Jordan representation ρ of M and a cyclic vector ξ for ρ. Then, denoting $r^\natural(\gamma) = \gamma \circ r$, $(r^\natural(\gamma)(\bar{\rho})\xi, \xi) = 0$ for all pair of normnal cyclic representation $\bar{\rho}$ of $W^*(M)$ and a cyclic vector ξ for $\bar{\rho}$. (Note that ρ and its canonical extension $\bar{\rho}$ have the same cyclic vector ξ.) Then, by the proof of Theorem 2.2, we have $r^\natural(\gamma) = \gamma \circ r = 0$. Noting $r(Q_H(W^*(M)_n) = Q_H(M)_n$, we conclude $\gamma = 0$. Hence j is injective.

By the same arguments as in the proof of Proposition 2.2, and upon considering the canonical extension, it is shown that the above definition of Jordan product is well defined, i.e., it does not depend on any particular CP-decompositions. We define a map $r^\natural : AB(Q_H(M)_n, B(H)_{sa}) \to AB(Q_H(W^*(M))_n, B(H)_{sa})$ by

$$r^\natural(\gamma) = \gamma \circ r \quad \text{for } \gamma \in AB(Q_H(M)_n, B(H)_{sa})$$

where r is the restriction map. Note here that, by the CP-duality for C*-algebras, $W^*(M)_{sa} \cong AB(Q_H(W^*(M))_n, B(H)_{sa})$ (Jordan isomorphism). Since r is onto, $r^\natural$ is an injective order isomorphism, hence a Jordan isomorphism with isometry $\|r^\natural(\gamma)\| = \|\gamma\|$, by which $\|\gamma\|$ defines the norm of JC-algebra $AB(Q_H(M)_n, B(H)_{sa})$.

It is straightforward to see that i is a Jordan isomorphism. (cf. Proof of

Theorem 2.2) ▮

The following theorems are the direct analogs of Theorems 2.4.A and B, and will be used in Chapter 7.

THEOREM 3.8.A. *Let A and B be JC-algebras, and H be a Hilbert space with* $dimH \geq \sup\{\alpha_c(A), \alpha_c(B)\}$. *Then,* $Q_H(A)$ *and* $Q_H(B)$ *are CP-affine BW-homeomorphic if and only if A ans B are Jordan isomorphic.*

THEOREM 3.8.B. *Let M and N be JW-algebras, and H be a Hilbert space with* $dimH \geq \sup\{\alpha_c(M)_n, \alpha_c(N)_n\}$. *Then,* $Q_H(M)_n$ *and* $Q_H(N)_n$ *are CP-affine isomorphic if and only if M and N are Jordan isomorphic.*

PROOF: We prove Part A. It is straightforward to see that, if A and B are Jordan isomorphic, then $Q_H(A)$ and $Q_H(B)$ are CP-affine BW-homeomorphic (cf. Proof of Theorem 2.4). Conversely, if $Q_H(A)$ and $Q_H(B)$ are CP-affine BW-homeomorphic, then $Q_{H,1}(A)$ and $Q_{H,1}(B)$ are w*-homeomorphic, hence by [51], A and B are Jordan isomorphic.

Part B is proved similarly. ▮

For our purpose in subsequent arguments, we denote by $P_H(A)$ the set of all pure elements of $CP(A, B(H)_{sa})$ for a JC-algebra A, and we prove the following result.

PROPOSITION 3.9. *Let A be a JC-algebra and H be a Hilbert space. Then,* $\psi = V^*\rho V \in P_H(A)$ *if and only if* $\rho \in JRep(A, H_\rho)$ *is irreducible. Moreover,* $P_H(A) = r(P_H(C^*(A)))$ *where r is the restriction map.*

PROOF: Let $\tilde{\psi} = V^*\tilde{\rho}V \in CP(C^*(A), B(H))$ be an extension of $\psi \in P_H(A)$ to $C^*(A)$. Since ψ has at least one pure extension, we can assume that $\tilde{\psi} \in P_H(C^*(A))$, i.e., $\tilde{\rho}$ is irreducible. We note that $\tilde{\rho}$ is irreducible if and only if ρ is irreducible since $C^*(A)$ is generated by A as a C*-algebra, so that ρ is irreducible.

Conversely, let $\psi = V^*\rho V \in CP(A, B(H)_{sa})$ with ρ irreducible, and assume $0 \le \varphi \le \psi$ with $\varphi \in CP(A, B(H)_{sa})$. We take the canonical extensions $0 \le \bar{\varphi} \le \bar{\psi} = V^*\bar{\rho}V$. Then $\bar{\psi} \in P_H(C^*(A))$ since $\bar{\rho}$ is irreducible, so $\bar{\varphi} = c\bar{\psi}$ $(0 \le c \le 1)$. Hence, $\varphi = r(\bar{\varphi}) = cr(\bar{\psi}) = c\psi$, i.e., $\psi \in P_H(A)$.

$P_H(A) = r(P_H(C^*(A)))$ is clear from the above proof. ▮

Remark. Let ρ be an irreducible representation of a JC-algebra A on a complex Hilbert space H. Then, $\overline{\rho(A)}^{w} \neq B(H)_{sa}$ (i.e., ρ is not a dense representation) in general, which is different from the C*-algebra case. Every irreducible representation is dense if and only if A is of complex type (cf. [4]).

CHAPTER 4

CP-FACIAL STRUCTURE OF CP-STATE SPACES AND DUALITY

In this chapter, we discuss the CP-affine geometry of CP-state spaces and generalize the duality between ideals of the algebra and faces of the state space, which was first formulated in scalar convexity by Effros [30] and Proser [63] for C*- and W*-algebras, and by Alfsen and Shultz [5], [7] for unital JB-and JBW-algebras. In the setting of CP-duality, the algebraic structure and order structure of the algebra, which are closely related, will be geometrically represented.

In scalar convexity theory, a face of a (convex) cone is characterized by a hereditary subcone. We will generalize this notion to the cone $CP(A, B(H))$ for a C*-algebra A with the completely positive order. We discuss the C*-algebra case first and feature the JC-algebra case later. It will be clarified how the C*-ideal structure is reduced to the non-oriented Jordan ideal structure through these parallel arguments.

DEFINITION 4.1. *Let A be a C*-algebra and H be a Hilbert space. A CP-convex cone C_H of $CP(A, B(H))$ is defined to be a CP-face, if it is CP-hereditary, i.e., if $\psi \in C_H$ and $\varphi \in CP(A, B(H)) \cap (\psi - CP(A, B(H)))$ imply $\varphi \in C_H$,* or *symbolically, $0 \leq \varphi \leq \psi \in C_H \Rightarrow \varphi \in C_H$. Accordingly, a CP-convex subset F_H of the CP-state space $Q_H(A)$ is defined to be a CP-face of $Q_H(A)$, if there exists a*

CP-face C_H of $CP(A, B(H))$ such that $F_H = C_H \cap Q_H(A)$.

A CP-face F_H is called a CP-split face of $Q_H(A)$ if there exists a CP-face F'_H of $Q_H(A)$, which is called the complementary CP-face of F_H, such that every CP-state $\psi \in Q_H(A)$ is decomposed uniquely as

$$\psi = \psi_1 + \psi_2 \quad \text{with } \psi_1 \in F_H \text{ and } \psi_2 \in F'_H,$$

i.e., $Q_H(A)$ is a CP-direct sum of F_H and F'_H, which will be written symbolically

$$Q_H(A) = F_H \oplus_{CP} F'_H.$$

For a subset $S \in Q_H(A)$, we will write $CP-face(S)$ for the CP-face generated by S. i.e., the smallest CP-face of $Q_H(A)$ containing S.

When we consider a W^*-algebra M and the cone $CP(M, B(H))_n$, the above definitions are used to define CP-face, CP-split face, $CP - face(S)$ of $Q_H(M)_n$ with the order induced from the cone $CP(M, B(H))_n$.

Remarks. 1. When $H = C$, the above definition of CP-face is reduced to the usual definition of face of the quasi-state space with respect to scalar convexity.

2. Let ψ be a point in $Q_H(A)$ and assume $\psi \neq 0$. Since $CP - face(\psi)$ need to be CP-convex, it has to contain at least the line segment $\{c\psi; 0 \leq c \leq 1\}$, so that we do not have the concept of "CP-extreme points" which are customary in the scalar theory. The origin $\{0\}$ is the only CP-face consisting of a single point.

3. To illustrate the difference between the concepts of face and CP-face, one can consider the pure ray of the cone $CP(A, B(H))$, i.e., $R_\psi = \{t\psi; \psi = V^*\pi V \in P_H(A)\ , t > 0\}$. As easily checked, R_ψ is CP-convex if and only if $dimH = 1$. As

a consequence, if $dim H \neq 1$, the pure line segment $[0, \psi]$ is an example which is a face of $Q_H(A)$ with respect to scalar convexity, but not a CP-face of $Q_H(A)$ with respect to CP-convexity.

We shall show that we can generalize the concept of projective faces in the scalar convexity theory ([5]) for our CP-convexity context.

THEOREM 4.2. *Let M be a W*-algebra and H be a Hilbert space. Then there exists a one to one correspondence between the set $\mathcal{P}(M)$ of all projections of M and the class $\mathcal{F}_H(M)$ of all norm closed CP-faces of $Q_H(M)_n$. The correspondence is given by*

$$p \in \mathcal{P}(M) \longleftrightarrow F_H(p) = \{\psi = V^*\pi V \in Q_H(M)_n; \psi(p) = V^*V\} \in \mathcal{F}_H(M)$$

where central projections are mapped onto CP-split faces.

PROOF: Let $p \in \mathcal{P}(M)$. We first show that $F_H(p)$ defined above is a norm closed CP-face of $Q_H(M)_n$. Set $q = e - p$ (e is the identity of M), then $F_H(p) = \{\psi \in Q_H(M)_n; \psi(q) = 0\}$. Set $C_H(q) = \{\psi \in CP(M, B(H))_n; \psi(q) = 0\}$. then $C_H(q)$ is a hereditary CP-convex cone such that $F_H(p) = Q_H(M)_n \cap C_H(q)$, so that $F_H(p)$ is a CP-face of $Q_H(M)_n$. Clearly $F_H(p)$ is norm closed, since it is the kernel of the norm continuous CP-affine function $q \in M \cong AB(Q_H(M)_n, B(H))$. Hence, $F_H(p) \in \mathcal{F}_H(M)$.

Conversely, suppose that F_H is a norm closed CP-face of $Q_H(M)_n$. Let $F_{\mathbf{C}}$ be the unitary equivalence classes of all one dimensional CP-maps in F_H, i.e., $F_{\mathbf{C}} = F_H \cap Q(M)_n := \hat{F}_{H,1}$, then $F_{\mathbf{C}}$ is a norm closed face of $Q(M)_n$, so that there exists

$q \in \mathcal{P}(M)$ such that $F_{\mathbf{C}} = \{\omega \in Q_H(M)_n; \omega(q) = 0\}$(cf.[60; Th. 3.6.11]). Observing the correspondence

$$\omega \in F_{\mathbf{C}} \leftrightarrow \hat{\psi}_\omega = (P\psi P)\hat{} = (\omega(\cdot)P)\hat{} \in \hat{F}_{H,1} \quad (\psi \in F_H)$$

where P is a one dimensional projection on H (cf. Proposition 1.5), we conclude that $\psi \in F_H$ if and only if $\psi(q) = 0$, hence $F_H = F_H(p)$ for $p = e - q \in \mathcal{P}(M)$.

We next consider the particular case where $c \in \mathcal{P}(M)$ is a central projection. Let $\psi = V^*\pi V \in Q_H(M)_n$ be arbitrary. Since $\pi(c) \in \pi(M)'$, we immediately have a decomposition

$$\psi = V^*\pi V = \psi_c + \psi_{e-c}$$

$$\text{where } \psi_c = V^*\pi(c)^*\pi(\cdot)\pi(c)V \quad \text{and} \quad \psi_{e-c} = V^*\pi(e-c)^*\pi(\cdot)\pi(e-c)V.$$

Clearly $\psi_c \in F_H(c)$ and $\psi_{e-c} \in F_H(e-c)$, since $\psi_c(e-c) = \psi_{e-c}(c) = 0$. We shall show that this is the unique decomposition of the form

$$\psi = \psi_1 + \psi_2; \quad \psi_1 \in F_H(c), \quad \psi_2 \in F_H(e-c).$$

In fact, since $\psi_1 \leq \psi$ and $\psi_2 \leq \psi$, by [11;Th.1.4.2] there exists $T \in \pi(M)'$ with $0 \leq T \leq I_{H_\pi}$ such that

$$\psi_1 = V^*T^{\frac{1}{2}}\pi(\cdot)T^{\frac{1}{2}}V \quad \text{and} \quad \psi_2 = V^*(I-T)^{\frac{1}{2}}\pi(\cdot)(I-T)^{\frac{1}{2}}V.$$

Then, noting that $\psi_1 \in F_H(c)$ and $\psi_2 \in F_H(e-c)$, we have

$$\psi_1(e-c) = V^*T^{\frac{1}{2}}\pi(e-c)T^{\frac{1}{2}}V = 0 \quad \text{and} \quad \psi_2(c) = V^*(I-T)^{\frac{1}{2}}\pi(c)(I-T)^{\frac{1}{2}}V = 0,$$

so that, for any $h \in H$,

$$(V^*T^{\frac{1}{2}}\pi(e-c)T^{\frac{1}{2}}Vh, h) = \|\pi(e-c)T^{\frac{1}{2}}Vh\|^2 = 0$$

and

$$(V^*(I-T)^{\frac{1}{2}}\pi(c)(I-T)^{\frac{1}{2}}Vh, h) = \|\pi(c)(I-T)^{\frac{1}{2}}Vh\|^2 = 0$$

i.e.,

$$\pi(e-c)T^{\frac{1}{2}}VH = 0 \quad \text{and} \quad \pi(c)(I-T)^{\frac{1}{2}}VH = 0.$$

Since $\pi(c) \in \pi(M) \cap \pi(M)'$ and $T \in \pi(M)'$, we conclude

$$\pi(e-c)T = \pi(c)(I-T) = 0 \quad \text{on} \;\; H_\pi = [\pi(M)VH],$$

from which we have $T = \pi(c)T = \pi(c)$ and $(I-T) = \pi(e-c)$ on H_π. This asserts $\psi_1 = \psi_c$ and $\psi_2 = \psi_{e-c}$.

To complete our proof, we have to show that every CP-split face is norm closed. Let F_H be a CP-split face of $Q_H(M)_n$. We define the map $\Psi_0 : \psi \in Q_H(M)_n \mapsto (P\psi P)\hat{} \in Q(M)_n := \hat{Q}_{H,1}(M)_n$, where P denotes any one dimensional projection on H, and $\hat{}$ denotes the unitary equivalence class. Then Ψ_0 is norm continuous, and maps F_H onto a split face $F_{\mathbf{C}} := \hat{F}_{H,1}$ in $Q(M)_n$. Since, in scalar convexity, every split face is norm closed (c.f.[43]), so $\Psi_0^{-1}(F_{\mathbf{C}}) = F_H$ is norm closed. This completes the proof. ∎

In [5], the projective face $F(p)$ of the normal state space $N(M)$ of a W*-algebra M was defined by $F(p) = \{\omega \in N(M); \omega(p) = 1\}$. In our setting of the quasi-state space, it is generalized as $F_{\mathbf{C}}(p) = \{\omega \in Q(M)_n; \omega(p) = \|\omega\|\}$. In the following, when we refer to the projective face in the scalar theory, we mean this generalized $F_{\mathbf{C}}(p)$. Corresponding to this, we call $F_H(p)$ in Theorem 4.2 a *CP-projective face*. For the convenience of our subsequent arguments, we summarize here some characterizations of $F_H(p)$.

PROPOSITION 4.3. *Under the same condition of Theorem 4.2,*

$$\begin{aligned}
F_H(p) &= \{\psi = V^*\pi V \in Q_H(M)_n;\ \psi(p) = V^*V\} \\
&= \{\psi = V^*\pi V \in Q_H(M)_n;\ im\, V \subset im\,(\pi(p)) = ker\,\pi(e-p)\} \\
&= \{\psi \in Q_H(M)_n; \psi(e-p) = 0\} \\
&= \{\psi \in Q_H(M)_n; (P\psi P)\check{} \in F_{\mathbf{C}}(p), P \text{ is a one dimensional projection on } H\} \\
&= \{\psi = \varphi_p; \varphi = V^*\pi V \in Q_H(M)_n,\ \ \varphi_p := V^*\pi(p)\pi(\cdot)\pi(p)V\}
\end{aligned}$$

PROOF: The first four equalities are immediate (cf. Proof of Theorem 4.2). For the last equality, we note that every $\psi = V^*\pi V \in F_H(p)$ can be written in the form $\psi = \psi_p = V^*\pi(p)\pi(\cdot)\pi(p)V$ since $\pi(p)\xi = \xi$ for all $\xi \in VH$, and that, for any $\varphi \in Q_H(M)_n$, $\varphi_p \in F_H(p)$ since $\varphi_p(e-p) = 0$. ∎

When we consider the CP-state space $Q_H(A)$ of a C*-algebra A, we can regard it as the normal CP-state space of the enveloping W*-algebra A^{**}, i.e., $Q_H(A) = Q_H(A^{**})_n$, and consider the CP-projective faces $F_H(p)$ for $p \in \mathcal{P}(A^{**})$. Then, the above characterizations of $F_H(p)$ are all applicable in this particular case. In what follows, when we make statements on the normal CP-state space of a W*-algebra, we always understand that this direct conversion to the CP-state space of a C*-algebra is implied.

COROLLARY 4.4. *Let A be a C*-algebra and H be a Hilbert space. Then, $F_H(p)$ with $p \in \mathcal{P}(A^{**})$ is BW-closed if and only if p is closed, i.e., there exists an assending net $(a_\alpha) \subset A^+$ such that $a_\alpha \nearrow p$ in the σ-weak topology in A^{**}. $p \in A$ if and only if $F_H(p)$ and $F_H(e-p)$ are both BW-closed.*

PROOF: We first show that $F_H(p)$ is BW-closed if and only if $F_{\mathbf{C}}(p)$ is w*-closed. By Proposition 4.3, we can consider the map $\Psi_0 : Q_H(M)_n \to Q(M)_n$, which is BW-w* continuous (cf. Proof of Theorem 4.2 for the definition of Ψ_0). If $F_H(p)$ is BW-closed, then $F_H(p)$ is BW-compact (cf.[11]), so $\Psi_0(F_H(p)) = F_{\mathbf{C}}(p)$ is w*-compact, hence w*-closed. Conversely, if $F_{\mathbf{C}}(p)$ is w*-closed, then $F_H(p) = \Psi_0^{-1}(F_{\mathbf{C}}(p))$ is BW-closed. Then the assertion follows from [1; Prop. II.2, Prop. II.18] or [5; Th. 11.7]. ∎

We shall now give some characterizations of CP-split faces.

PROPOSITION 4.5. *Let M be a W^*-algebra and H be a Hilbert space, and let $F_H(p)$ be a CP-projective face of $Q_H(M)_n$. Then, $F_H(p)$ is a CP-split face if and only if one of the following equivalent conditions is satisfied.*

- *(i)* *p is a central projection.*
- *(ii)* *For any $\psi = V^*\pi V \in F_H(p)$, $\pi(p) = I_{H_\pi}$.*
- *(iii)* *For any $\psi = V^*\pi V \in F_H(p)$, and any unitary element $u \in M$,*

$$\psi_u = V^*\pi(u^*)\pi(\cdot)\pi(u)V \in F_H(p).$$

PROOF: From Theorem 4.2, $F_H(p)$ is a split face if and only if p is central. We will show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) : Let $\psi = V^*\pi V \in F_H(p)$. Then, letting $q = e - p$, we have $V^*\pi(q)V = 0$, so that $\pi(q)VH = 0$. Since q is central, for any $a \in M$, $\pi(q)\pi(a)VH = \pi(a)\pi(q)VH = 0$, so $\pi(q)[\pi(a)VH] = \pi(q)H_\pi = 0$, i.e., $\pi(q) = 0$. Thus, $\pi(p) = \pi(e) - \pi(q) = I_{H_\pi}$.

(ii) ⇒ (iii) : Let $\psi = V^*\pi V \in F_H(p)$ and assume $\pi(p) = I_{H_\pi}$. Then, $\psi_u(p) = V^*\pi(u)^*\pi(p)\pi(u)V = V^*\pi(u^*u)V = V^*V$, hence $\psi_u \in F_H(p)$.

(iii) ⇒ (i) : Let $F_{\mathbf{C}}(p) := \hat{F}_{H,1}(p)$, then $F_{\mathbf{C}}(p)$ is the projective face of $Q(M)_n$ corresponding to $F_H(p)$ (Proposition 4.3). Let $\psi = V^*\pi V \in F_H(p)$. Then, by Proposition 1.5, for any one dimensional projection P on H,

$$P\psi P = PV^*\pi_\omega VP = \omega(\cdot)P \ \in F_{H,1}(p) \quad \text{with } \omega \in F_{\mathbf{C}}(p)$$

where $\pi_\omega = \pi|_{[\pi(M)VPH]}$ is a cyclic representation corresponding to ω. Then, by assumption (iii),

$$\omega(u^*(\cdot)u)P = PV^*\pi_\omega(u)^*\pi_\omega(\cdot)\pi_\omega(u)VP \in F_{H,1}(p).$$

Considering the equivalence class, we conclude that $F_{\mathbf{C}}(p)$ is invariant under the operation

$$\omega \in F_{\mathbf{C}}(p) \longmapsto \omega(u^*(\cdot)u) \in F_{\mathbf{C}}(p).$$

By [2; Prop.A.2], it follows that $F_{\mathbf{C}}(p)$ is a split face of $Q(M)_n$, hence p is a central projection. This completes the proof. ∎

The following CP-split decompositions will be used in the later chapters.

COROLLARY 4.6. *Let M be a W^*-algebra and H be a Hilbert space. Then we have the following CP-direct sum decompositions.*

(i) $Q_H(M) = Q_H(M)_n \oplus_{CP} Q_H(M)_s$ *where* $Q_H(M)_s := (Q_H(M)_n)'$.

(ii) $Q_H(M)_n = Q_H(M)_n^e \oplus_{CP} Q_H(M)_n^{ne}$ *where* $Q_H(M)_n^e := CP-conv$ $P_H(M)_{n,1}$ *with* $P_H(M)_{n,1} = P_H(M) \cap Q_H(M)_n$ *and* $Q_H(M)_n^{ne} := (Q_H(M)_n^e)'$.

(iii) $Q_H(M)_n = Q_H(M)_n^{ab} \oplus_{CP} Q_H(M)_n^{nab}$ where $Q_H(M)_n^{ab} := \{\psi = V^*\pi V \in Q_H(M)_n; \pi$ *is of type* $I_1\}$ and $Q_H(M)_n^{nab} := \{\psi = V^*\pi V \in Q_H(M)_n; \pi$ *has no type* I_1 *part*$\}$.

PROOF: (i) : Let $\psi = V^*\pi V \in Q_H(M)_n$. Let $\tilde{\pi}$ be the canonical extension of π to M^{**}, and let $\tilde{\psi} := V^*\tilde{\pi}V \in Q_H(M) = Q_H(M^{**})_n$. Then, for any unitary $u \in M^{**}$, $\tilde{\pi}(u)^*\tilde{\pi}\tilde{\pi}(u)$ is normal for M, so $\tilde{\psi}_u = V^*\tilde{\pi}(u)^*\tilde{\pi}\,\tilde{\pi}(u)V \in Q_H(M)_n$. Hence, by Proposition 4.5 (iii), $Q_H(M)_n$ is a split face of $Q_H(M)$. $Q_H(M)_s = (Q_H(M)_n)'$ is the complementary face of $Q_H(M)_n$.

(ii) : Let $\psi \in Q_H(M)_n^e$, i.e., ψ can be written $\psi = \sum_\alpha S_\alpha^*\psi_\alpha S_\alpha$ where $\psi_\alpha = V_\alpha^*\pi_\alpha V_\alpha \in P_H(M)_{n,1}$ and $S_\alpha \in B(H, H_{\pi_\alpha})$ such that $\sum_\alpha S_\alpha^* S_\alpha \leq I_H$. Then, for any unitary $u \in M$, $(\psi_\alpha)_u = V_\alpha^*\pi_\alpha(u)^*\pi_\alpha\,\pi_\alpha(u)V_\alpha \in P_H(M)_{n,1}$ since $\pi_\alpha(u)^*\pi_\alpha\,\pi_\alpha(u) \in Irr(M)_n$. Hence $\psi_u = \sum_\alpha S_\alpha(\psi_\alpha)_u S_\alpha \in Q_H(M)_n^e$, and we can apply Proposition 4.5 (iii).

(iii) : Let $\psi = V^*\pi V \in Q_H(M)_n^{ab}$, i.e., π is of type I_1. Then, for any unitary $u \in M$, $\pi(u)^*\pi\,\pi(u)$ is of type I_1, so we apply Proposition 4.5 (iii). ▮

DEFINITION 4.7. *In the CP-direct sum decomposition of Corollary 4.6, we define the CP-split faces*

(i) $Q_H(M)_n$ *[resp.* $Q_H(M)_s$*] to be the* ***normal*** *[resp.* ***singular*** *] part.*

(ii) $Q_H(M)_n^e$ *[resp.* $Q_H(M)_n^{ne}$*] to be the* ***elementary*** *[resp.* ***non-elementary*** *] part.*

(iii) $Q_H(M)_n^{ab}$ *[resp.* $Q_H(M)_n^{nab}$*] to be the* ***abelian*** *[resp.* ***non-abelian*** *] part.*

Remark. The elementary [resp. non-elementary] part in the normal CP-state spaces in (ii) corresponds to the ***atomic*** [resp. ***non-atomic***] part in the state spaces. (i) and (iii) are direct analogs of the scalar theory.

We next consider the duality between the ideal structure of the algebra and the CP-facial structure of the CP-state space.

DEFINITION 4.8. *Let* A *be a* C^**-algebra,* M *be a* W^**-algebra, and* H *be a Hilbert space. We use the following notations for the annihilators of subsets* $B \subset A$, $N \subset M$, $F \subset Q_H(A)$ *and* $G \subset Q_H(M)_n$ *;*

$$B^{\perp} := \{\psi \in Q_H(A); \psi(a) = 0 \quad \text{for all } a \in B\}$$

$$N_{\perp} := \{\psi \in Q_H(M)_n; \psi(a) = 0 \quad \text{for all } a \in N\}$$

$${}_{\perp}F := \{a \in A; \psi(a) = 0 \quad \text{for all } \psi \in F\}$$

$${}^{\perp}G := \{a \in M; \psi(a) = 0 \quad \text{for all } \psi \in G\}$$

(Note the place of $\perp$*, e.g.,* $\{p\}_{\perp} = F_H(e-p)$ *for* $p \in \mathcal{P}(M)$*.)*

We also define, for any projective face $F_H(p)$*, the left [resp. right, and two sided] annihilators of* $p \in M$*;*

$${}^{\perp}F_H(p)^l := \{a \in {}^{\perp}F_H(p);\ ap = 0\} = \{a \in M;\ ap = 0\}$$

$${}^{\perp}F_H(p)^r := \{a \in {}^{\perp}F_H(p);\ pa = 0\} = \{a \in M;\ pa = 0\}$$

$${}^{\perp}F_H(p)^{lr} := \{a \in {}^{\perp}F_H(p);\ ap = pa = 0\} = \{a \in M;\ ap = pa = 0\}$$

Similarly, we define ${}_{\perp}F_H(p)^l$, ${}_{\perp}F_H(p)^r$ *and* ${}_{\perp}F_H(p)^{lr}$ *as subsets of* A *for* $p \in \mathcal{P}(A^{**})$.

Recall that every weak*-closed left [resp. right] ideal I of a W*-algebra M is of the form $I = Mp$ [resp. $I = pM$] for some $p \in \mathcal{P}(M)$, and every weak*-closed order ideal J of M is of the form $J = pMp$ with $p \in \mathcal{P}(M)$. (This is implied in [30], [60]. Here we call $J \subset M$ an order ideal of M if J_{sa} is an order ideal of M_{sa} in the order M^+). The following theorem gives the duality between these ideals and the CP-projective faces $F_H(p)$.

THEOREM 4.9. *Let M be a W*-algebra and H be a Hilbert space with $dim H \geq \alpha_c(M)_n$. Then there exist one to one correspondences between the weak*-closed left [resp. right] ideals Mp [resp. pM], the weak*-closed order ideals pMp and the projective faces $F_H(p)$ such that*

$$F_H(e-p) = (Mp)_\perp = (pM)_\perp = (pMp)_\perp$$

and

$$^\perp F_H(e-p) = Mp + pM.$$

More specifically,

$$\begin{aligned} Mp &= {}^\perp F_H(e-p)^l \\ pM &= {}^\perp F_H(e-p)^r \\ pMp = Mp \cap pM &= {}^\perp F_H(e-p)^{lr} \\ &\cong AB(F_H(p), B(H)) \quad (\text{*-isomorphism}). \end{aligned}$$

If p is a central projection, then the above three ideals coincide as the weak-closed two sided ideal $Mp = pM = pMp$, which corresponds to the CP-split face $F_H(p)$.*

Proof: It is obvious that $(Mp)_\perp \subset \{p\}_\perp = F_H(e-p)$. Conversely, if $\psi = V^*\pi V \in F_H(e-p)$, then, by Proposition 4.3, $im\, V \subset ker\, \pi(p)$, so $\psi(ap) = V^*\pi(a)\pi(p)V = 0$ for all $a \in M$, i.e., $\psi \in (Mp)_\perp$. Hence, $F_H(e-p) = (Mp)_\perp$. Similarly, we can show $F_H(e-p) = (pM)_\perp = (pMp)_\perp$.

It is clear that ${}^\perp F_H(e-p) \supset Mp + pM$. Conversely, let $a \in {}^\perp F_H(e-p)$. By Proposition 4.3, $\psi \in F_H(e-p)$ is expressed as $\psi = \varphi_{e-p} = V^*\pi(e-p)\pi(\cdot)\pi(e-p)V$ for $\varphi = V^*\pi V \in Q_H(M)_n$, so we have

$$\psi(a) = \varphi_{e-p}(a) = V^*\pi(e-p)\pi(a)\pi(e-p)V = V^*\pi((e-p)a(e-p))V = 0.$$

Since $\varphi = V^*\pi V \in Q_H(M)_n$ is arbitrary, by the CP-duality theorem for M (Theorem 2.2.B), we conclude

$$(e-p)a(e-p) = 0.$$

Hence, upon introducing $a_1 := ap - \frac{1}{2}pap$ and $a_2 := pa - \frac{1}{2}pap$, we have

$$a_1(e-p) = a_1 - a_1 p = 0 \quad \text{and} \quad (e-p)a_2 = a_2 - pa_2 = 0,$$

i.e.,

$$a = a_1 + a_2 \quad \text{with } a_1 = a_1 p \text{ and } a_2 = pa_2,$$

from which we conclude $a \in Mp + pM$. Hence, ${}^\perp F_H(e-p) = Mp + pM$.

The equalities $Mp = {}^\perp F_H(e-p)^l, pM = {}^\perp F_H(e-p)^r$ and $pMp = {}^\perp F_H(e-p)^{lr}$ are immediate from the definition.

To show the *-isomorphism $pMp \cong AB(F_H(p), B(H))$, we prove that the evaluation map

$$i : pMp \to AB(F_H(p), B(H)),$$

is bijective.

We first show that i is injective. Indeed, if $\varphi_p(pap) = 0$ for all $\varphi_p \in F_H(p)$ with $\varphi = V^*\pi V \in Q_H(M)_n$, then $V^*\pi(p)\pi(pap)\pi(p)V = V^*\pi(pap)V = \varphi(pap) = 0$ for all $\varphi \in Q_H(M)_n$, hence $pap = 0$.

We now show that i is surjective. Define

$$\Phi_H(p) : \varphi = V^*\pi V \in Q_H(M)_n \mapsto \varphi_p = V^*\pi(p)\pi(\cdot)\pi(p)V \in F_H(p),$$

which is obviously a CP-affine map, and define its CP-dual

$$\Phi_H(p)^\natural : \gamma \in AB(F_H(p), B(H)) \longmapsto \gamma \circ \Phi_H(p) \in AB(Q_H(M)_n, B(H)).$$

We note that $\Phi_H(p)^\natural$ is injective, since $\Phi_H(p)$ is surjective (cf. Proposition 4.3). From the CP-duality $M \cong AB(Q_H(M)_n, B(H))$ (Theorem 2.2), we can set $\Phi_H(p)^\natural(\gamma) = \gamma \circ \Phi_H(p) = m_\gamma \in M$. Then, for any $\varphi \in Q_H(M)_n$,

$$\begin{aligned}\Phi_H(p)^\natural(\gamma)(\varphi) &= (\gamma \circ \Phi_H(p))(\varphi) = (\gamma \circ \Phi_H(p)^2)(\varphi) \\ &= (\gamma \circ \Phi_H(p))(\Phi_H(p)(\varphi)) = \Phi_H(p)^\natural(\gamma)(\Phi_H(p)(\varphi)),\end{aligned}$$

and letting here $\varphi = V^*\pi V$, we have

$$V^*\pi(m_\gamma)V = V^*\pi(p)\pi(m_\gamma)\pi(p)V,$$

so that

$$\varphi(m_\gamma) = \varphi(pm_\gamma p) \quad \text{for all } \varphi \in Q_H(M)_n,$$

which implies $m_\gamma = pm_\gamma p$. Hence, for any $\psi \in F_H(p)$, noting that $\psi = \Phi_H(p)\psi$,

$$\gamma(\psi) = \gamma(\Phi_H(p)(\psi)) = \psi(m_\gamma) = \psi(pm_\gamma p) = i(pm_\gamma p)(\psi).$$

Hence i is surjective.

If p is a central projection, then $Mp = pM = pMp$ is a two sided ideal, and by Theorem 4.2, $F_H(p)$ is a CP-split face. This completes the proof. ∎

Remark. It follows from the above isomorphism that there exists one to one correspondence between the idenpotent maps

$$\Phi_H(p) : \varphi \in Q_H(M)_n \longmapsto \varphi_p \in F_H(p)$$

$$\Phi_H(p)^\natural : m \in M \longmapsto pmp \in pMp$$

and the images $\Phi_H(p)(Q_H(M)_n) = F_H(p)$ and $\Phi_H(p)^\natural(M) = pMp$ are in the CP-duality

$$pMp \cong AB(F_H(p), B(H)) \quad (\text{*-isomorphism}),$$

which generalizes Alfsen-Shultz's duality in [7;Prop. 1.10].

Let I be a left [resp. right] ideal of a C*-algebra A, then $I^* := \{a^*; a \in I\}$ is a right [resp. left] ideal of A, and $I \cap I^*$ is an order ideal of A. The following is the C*-version of Theorem 4.9.

THEOREM 4.10. *Let A be a C*-algebra and H be a Hilbert space with $dim H \geq \alpha_c(A)$. Then, there exists one to one correspondences between the closed left [resp. right] ideals I [resp. I^*] of A, the closed order ideals $I \cap I^*$, and BW-closed CP-projective faces $F_H(q)$ with closed $q \in \mathcal{P}(A^{**})$ such that*

$$F_H(q) = I^\perp = (I^*)^\perp = (I \cap I^*)^\perp \qquad \text{and} \qquad {}_\perp F_H(q) = I + I^*$$

More specifically,

$$I = {}_\perp F_H(q)^l, \qquad I^* = {}_\perp F_H(q)^r \qquad \text{and} \qquad I \cap I^* = {}_\perp F_H(q)^{lr}.$$

If $p = e - q \in A$, *then we have*

$$I \cap I^* = pAp \cong AC(F_H(p), B(H)) \qquad (\text{*-isomorphism})$$

In particular, each closed two sided ideal J *corresponds to the BW-closed CP-split face* $F_H(q)$ *with central projection* $q \in \mathcal{P}(A^{**})$ *such that*

$$F_H(q) = J^{\perp} \qquad \text{and} \qquad {}_{\perp}F_H(q) = J$$

and we have the isomorphisms, with $p = e - q$,

$$J \cong AC(F_H(p), B(H)) \qquad \text{and} \qquad A/J \cong AC(F_H(q), B(H)).$$

PROOF: Let I be a closed left ideal of A. Clearly the face $I^{\perp}$ is BW-closed, hence by Corollary 4.4, there corresponds a closed $q \in \mathcal{P}(A^{**})$ such that $I^{\perp} = F_H(q)$. Clearly, $(I^*)^{\perp} = I^{\perp} = F_H(q)$. It is obvious that $(I \cap I^*)^{\perp} \supset I^{\perp}$. To see the converse inclusion, if $\psi = 0$ on $I \cap I^*$ for $\psi \in Q_H(A)$, then $\psi = 0$ on the left ideal generated by $I \cap I^*$, which equals I. Hence, $(I \cap I^*)^{\perp} = I^{\perp} = F_H(q)$.

It is obvious that ${}_{\perp}F_H(q) \supset I + I^*$. On the other hand, by Proposition 4.9, ${}^{\perp}F_H(q) = A^{**}p + pA^{**}$ with $p = e - q$, so ${}_{\perp}F_H(q) = A^{**}p \cap A + pA^{**} \cap A$. Set $I' = A^{**}p \cap A$, then I' is a left ideal of A, and clearly $I' \supset I$. Noting that $I^{\perp} = F_H(q) = (A^{**}p)_{\perp} \subset (I')^{\perp} \subset I^{\perp}$, we have $(I')^{\perp} = I^{\perp}$ so that $I' = I$ (by Hahn-Banach's theorem). We proved $A^{**}p \cap A = I$ and $pA^{**} \cap A = I^*$, so that ${}_{\perp}F_H(q) = I + I^*$.

It is now straightforward, from definition, that $I = {}_{\perp}F_H(q)^l$ and $I^* = {}_{\perp}F_H(q)^r$, and $I \cap I^* = {}_{\perp}F_H(q)^{lr}$.

Assume $p = e - q \in A$, or equivallently both $F_H(p)$ and $F_H(q)$ are BW-closed (cf. Corollary 4.4). Then, from the arguments above, $I \cap I^* = (A^{**}p \cap pA^{**}) \cap A = pA^{**}p \cap A = pAp$. We have already shown, in Theorem 4.9, $pA^{**}p \cong AB(F_H(p), B(H))$ (*-isomorphism). Obviously, with pAp considered as a subset of $pA^{**}p$, the evaluation map

$$i : I \cap I^* = pAp \to AC(F_H(p), B(H))$$

is injective. We claim that i is surjective. We note that $\Phi_H(p) : \varphi \in Q_H(A) \to \varphi_p \in F_H(p)$ is a BW-continuous CP-affine map since $p \in A$, so that, for $\gamma \in AC(F_H(p), B(H))$, $\Phi_H(p)^\natural(\gamma) = \gamma \circ \Phi_H(p) \in AC(Q_H(A), B(H))$. Hence, from the CP-duality $A \cong AC(Q_H(A), B(H))$ (Theorem 2.2), by the same arguments as in the proof of Theorem 4.9, we can set $\Phi_H(p)^\natural(\gamma) \in m_\gamma \in A$, and we can show that $\gamma(\psi) = i(pm_\gamma p)(\psi)$ for all $\psi \in F_H(p)$, which proves that i is surjective.

If J is a two sided ideal, then $J = J^*$, so $F_H(q) = J^\perp$ and ${}_\perp F_H(q) = J$. By bipolar theorem, ${}^\perp F_H(q) = J^{\circ\circ} = \sigma(A^{**}, A^*) - cl.\, J \cong J^{**}$. Since J is a two sided ideal, so is ${}^\perp F_H(q)$, then by Theorem 4.9, $F_H(q)$ is a CP-split face. Further, by Theorem 4.9, ${}^\perp F_H(q) = A^{**}p \cong AB(F_H(p), B(H))$ with $p = e - q$. Hence $F_H(p)$ is identified with $Q_H(J^{**})_n = Q_H(J)$, and we have $J \cong AC(F_H(p), B(H))$ where we equip $F_H(p)$ with the BW-topology defined for J. Clearly, $A/J \cong AC(Q_H(A), B(H))/{}_\perp F_H(q) \cong AC(F_H(q), B(H))$. This completes the proof. ∎

We can generalize our earlier observation of the duality between the ideal structure of C*- and W*-algebras and CP-facial structure of the CP-state spaces to JC-algebras. We employ the same definitions and notations as in the C*-algebra case to define the corresponding concepts for JC-algebras. We however need to introduce the concept of *(norm) exposed* CP-face. A norm closed CP-face F_H of a CP-state space is defined to be (norm) exposed if $F_{\mathbf{C}} := \hat{F}_{H,1}$ is a (norm) exposed face of the corresponding state space.

For the standard theory of JB-algebras, we refer to [5], [7], [8], [45], and [67]; the essential features of this theory are summarized in Appendices 3 and 5. Recall that a *P-projection* U_p for a JBW-algebra M is an idempotent map defined by $U_p : a \in M \mapsto \{pap\} = 2p \circ (p \circ a) - p \circ a$ for all $a \in M$ where p is an idempotent element of M (i.e., $p^2 = p$), which satisfies $im\, U_p = ker\, U_{e-p}$ and $ker\, U_p = im\, U_{e-p}$. If the algebra is special, then the Jordan triple product $\{pap\}$ is reduced to the operator product pap, so that $U_p(a) = pap$ for all $a \in M$. For a JBW-algebra M, the projective face $F_{\mathbf{C}}(p)$ in the quasi state space $Q(M)_n$ is given by $F_{\mathbf{C}}(p) = im\, U_p^* \cap Q(M)_n = \{\omega \in Q(M)_n ; \omega(p) = \|\omega\|\}$.

We first generalize Theorem 4.2 as follows.

THEOREM 4.11. *Let M be a JW-algebra and H be a Hilbert space. Then there exists a one to one correspondence between the set $\mathcal{U}(M)$ of all idempotent elements of M and the class $\mathcal{F}_H(M)$ of all norm exposed CP-faces of $Q_H(M)_n$. The correspondence is given by*

$$p \in \mathcal{U}(M) \longleftrightarrow F_H(p) = \{\psi = V^*\rho V \in Q_H(M)_n ; \psi(p) = V^*V\} \in \mathcal{F}_H(M)$$

where central idempotent elements are mapped onto CP-split faces.

PROOF: It is straightforward that $F_H(p)$ is a norm exposed CP-face of $Q_H(M)_n$ since the corresponding one dimensional face $F_{\mathbf{C}} := \hat{F}_{H,1}(p)$ is norm exposed (cf. [5;Prop. 2.15]). Conversely, if F_H is a norm exposed CP-face of $Q_H(M)_n$, then $F_{\mathbf{C}}$ is a norm exposed face of $Q(M)_n$, hence by [5; Prop. 6.2] it is projective, i.e., there exists an idenpotent element $p \in \mathcal{U}(M)$ such that $F_{\mathbf{C}} = \{\omega \in Q(M)_n; \omega(e-p) = 0\}$, from which it follows that $F_H = \{\psi \in Q_H(M)_n; \psi(e-p) = 0\} = F_H(p)$.

Next we assume that $p = c$ is central (cf. Appendix 3). Then any $\psi = V^*\rho V \in Q_H(M)_n$ is decomposed as

$$\psi = \psi_c + \psi_{e-c} \quad \text{where} \quad \psi_c = V^*\rho(c) \circ \rho(\cdot)V \in F_H(c)$$

$$\text{and} \quad \psi_{e-c} = V^*\rho(e-c) \circ \rho(\cdot)V \in F_H(e-c).$$

We will show that this decomposition is unique. Indeed, let $\psi = \psi_1 + \psi_2$ be another decomposition with $\psi_1 \in F_H(c)$ and $\psi_2 \in F_H(e-c)$. Then, for any one dimensional projection P, $(P\psi_1 P)\hat{} \in F_{\mathbf{C}}(p)$ and $(P\psi_2 P)\hat{} \in F_{\mathbf{C}}(e-c)$. We note (cf. [5]) that $F_{\mathbf{C}}(c) = im\, U_c^* \cap Q(M)_n = ker\, U_{e-c}^* \cap Q(M)_n$ and $F_{\mathbf{C}}(e-c) = im\, U_{e-c}^* \cap Q(M)_n = ker\, U_c^* \cap Q(M)_n$ where U_c is the P-projection corresponding to $c \in \mathcal{U}(M)$, i.e., $U_c(a) = c \circ a$ for all $a \in M$. We then have

$$U_c^*(P\psi P) = U_c^*(P\psi_1 P) + U_c^*(P\psi_2 P) = P\psi_1 P$$

$$U_{e-c}^*(P\psi P) = U_{e-c}^*(P\psi_1 P) + U_{e-c}^*(P\psi_2 P) = P\psi_2 P.$$

Hence, for any $a \in M$,

$$P\psi_1(a)P = P\psi(U_c(a))P = P\psi(c \circ a)P = PV^*\rho(c) \circ \rho(a)VP$$

$$P\psi_2(a)P = P\psi(U_{e-c}(a))P = P\psi((e-c) \circ a)P = PV^*\rho(e-c) \circ \rho(a)VP.$$

Since $a \in M$ is arbitrary and P is any one dimensional projection, we have

$$\psi_1 = V^*\rho(c) \circ \rho(\cdot)\, V \qquad \text{and} \qquad \psi_2 = V^*\rho(e-c) \circ \rho(\cdot)\, V$$

which proves the uniqueness of the decomposition.

We note that every CP-split face F_H is norm exposed since its corresponding one dimensional face $F_{\mathbf{C}} = \hat{F}_{H,1}$ is a split face, so norm exposed. This completes the proof. ▮

We have characterizations of CP-projective faces similar to those proved in Proposition 4.3 in the C*-algebra case.

PROPOSITION 4.12. *Under the condition of Theorem 4.11,*

$$\begin{aligned} F_H(p) &= \{\psi = V^*\rho V \in Q_H(M)_n;\, \psi(p) = V^*V\} \\ &= \{\psi = V^*\rho V \in Q_H(M)_n;\, im\, V \subset im\,(\rho(p)) = ker\,(\rho(e-p))\} \\ &= \{\psi \in Q_H(M)_n; \psi(e-p) = 0\} \\ &= \{\psi \in Q_H(M)_n; (P\psi P)\hat{} \in F_{\mathbf{C}}(p), P \text{ is a one dimensional projection on } H\} \\ &= \{\psi = \varphi_p;\ \varphi = V^*\rho V \in Q_H(M)_n,\ \varphi_p := V^*\rho \circ U_p\, V = V^*\rho(p)\rho(\cdot)\rho(p)\, V\} \end{aligned}$$

PROOF: Since the proof goes in parallel as in Theorem 4.3, we just prove the last equality. We note that every $\psi = V^*\rho V \in F_H(p)$ can be written in the form $\psi = \psi_p = V^*\rho \circ U_p\, V$. In fact, since $\rho(M)$ is special, $\rho \circ U_p(a) = \rho(\{pap\}) = \rho(p)\rho(a)\rho(p)$ for all $a \in M$, and since $\rho(p)\xi = \xi$ for all $\xi \in VH$, we have $\psi_p(a) = V^*\rho(p)\rho(a)\rho(p)V = V^*\rho(a)V = \psi(a)$ for all $a \in M$. Hence, $\psi = \psi_p$. Clearly $\varphi_p \in F_H(p)$ for all $\varphi \in Q_H(M)_n$ since $\varphi_p(e-p) = 0$. ▮

Now, Proposition 4.5 is generalized as follows.

PROPOSITION 4.13. *Let M be a JW-algebra and H be a Hilbert space, and let $F_H(p)$ be a CP-projective face. Then, $F_H(p)$ is a CP-split face of $Q_H(M)_n$ if and only if one of the following equivalent conditions is satisfied.*

(i) *p is a central idempotent.*

(ii) *For any $\psi = V^*\rho V \in F_H(p)$, $\rho(p) = I_{H_\rho}$.*

(iii) *For any $\psi = V^*\rho V \in F_H(p)$, and for any symmetry $s \in M$ (i.e., $s^2 = e$),*

$$\psi_s = V^*(\rho \circ U_s) V = V^*\rho(s)\rho(\cdot)\rho(s) V \in F_H(p)$$

PROOF: From Theorem 4.11, $F_H(p)$ is a split face if and only if p is a central element.

(i) ⇒ (ii) is shown by the same argument as that used in Proposition 4.5.

(ii) ⇒(iii) : Let $\psi = V^*\rho V \in F_H(p)$. Since $\rho(p) = I_{H_\rho}$,

$$\begin{aligned} \psi_s(p) &= V^*\rho(U_s(p)) V = V^*\rho(\{sps\}) V = V^*\rho(s)\rho(p)\rho(s) V \\ &= V^*\rho(s^2) V = V^*\rho(e) V = V^*V. \end{aligned}$$

(iii) ⇒ (i) : Assume $F_H(p)$ is invariant under the operation $\psi = V^*\rho V \mapsto \psi_s = V^*(\rho \circ U_s) V$ for every symmetry s. Then, $F_{\mathbf{C}}(p) = \hat{F}_{H,1}(p)$ is invariant under the operation

$$\omega \in F_{\mathbf{C}}(p) \longmapsto \omega(U_s(\cdot)) \in F_{\mathbf{C}}(p),$$

so that

$$U_s^*(im\, U_p^*) \subset im\, U_p^* \qquad \text{and} \qquad U_s(im\, U_p^*)^\circ \subset (im\, U_p^*)^\circ.$$

Since $(im\, U_p^*)^\circ = ker\, U_p = im\, U_{e-p}$, the last inclusion implies $U_s(im\, U_{e-p}) \subset im\, U_{e-p}$. We note the duality $im\, U_p \cong (im\, U_p^*)^*$ ([7;Prop. 1.10]), and the fact that U_s is a positive map (e.g.[8;Prop.2.7]) with $\|U_s\| = 1$ (since $U_s(e) = e$), we conclude

$$U_s^*(F_{\mathbf{C}}(p)) \subset F_{\mathbf{C}}(p) \qquad \text{and} \qquad U_s^*(F_{\mathbf{C}}(e-p)) \subset F_{\mathbf{C}}(e-p)$$

We note here that, since s is a symmetry, U_s is an automorphism (e.g.,[45; 2.8.6]), so $U_s(p)$ is an idempotent. Hence,

$$\omega(U_s(p)) = 1 \qquad \text{for all } \omega \in F_{\mathbf{C}}(p)$$

$$\omega(U_s(p)) = 0 \qquad \text{for all } \omega \in F_{\mathbf{C}}(e-p)$$

Recall from [5] that

$$p = \inf\{a \in M; 0 \le a \le e,\ \ a = 1 \ \text{ on } F_{\mathbf{C}}(p)\}$$

$$= \sup\{b \in M; 0 \le b \le e;\ \ b = 0 \ \text{ on } F_{\mathbf{C}}(e-p)\},$$

from which we conclude $U_s(p) \ge p$ and $U_s(p) \le p$, i.e., $U_s(p) = p$. This implies that p is invariant under U_s for all symmetry $s \in M$, which is equivalent to asserting that p is central ([8;Lemma 5.3]). ∎

COROLLARY 4.14. *Let M be a JW-algebra and H be a Hilbert space. Then, we have the following CP-split decompositions.*

(i) $Q_H(M) = Q_H(M)_n \oplus_{CP} Q_H(M)_s$ *where* $Q_H(M)_s := (Q_H(M)_n)'$.

(ii) $Q_H(M)_n = Q_H(M)_n^e \oplus_{CP} Q_H(M)_n^{ne}$ *where* $Q_H(M)_n^e := CP-conv\ P_H(M)_{n,1}$ *with* $P_H(M)_{n,1} = P_H(M) \cap Q_H(M)_n$ *and* $Q_H(M)_n^{ne} := (Q_H(M)_n^e)'$.

PROOF: The arguments are parallel to those used in Corollary 4.6.

(i) : We just note that, for any $\psi = V^*\rho V \in Q_H(M)_n$, $\tilde{\psi}_s = V^*\tilde{\rho}(s)\tilde{\rho}\tilde{\rho}(s)V \in Q_H(M)_n$ for all symmetry $s \in M^{**}$, where $\tilde{\psi}$ and $\tilde{\rho}$ denote the canonical extensions of ψ and ρ to A^{**}. We then apply Proposition 4.13 (iii).

(ii) : For any symmetry $s \in M$, U_s is a Jordan automorphism. Then, ρ is irreducible if and only if $\rho \circ U_s$ is irreducible. Hence, by Prop. 3.9, $Q_H(M)_n^e$ is U_s^* invariant. ∎

DEFINITION 4.15. *According to the CP-direct sum decomposition of Corollary 4.14, we define*

(i) $Q_H(M)_n$ *[resp.* $Q_H(M)_s$*] to be the* ***normal*** *[resp.* ***singular*** *] part.*

(ii) $Q_H(M)_n^e$ *[resp.* $Q_H(M)_n^{ne}$*] to be the* ***elementary*** *[resp.* ***non-elementary*** *] part.*

Remark. The normal-singular decomposition of the dual spaces of JBW-algebras, and, as its applications, the topological properties of the dual spaces of JBW-algebras were studied in [36].

The following theorem is a direct extension of Alfsen-Shultz [7; Prop.1.10] and Shultz [67;Lemma 2.1].

THEOREM 4.16. *Let M be a JW-algebra, H be a Hilbert space with* $dim H \geq \alpha_c(M)_n$ *and let* $p \in \mathcal{U}(M)$. *Then there exists one to one correspondence between the weak*-closed order ideal* $U_p(M) = \{pMp\}$ *and the projective face* $F_H(p)$ *such that*

$$F_H(e-p) = U_p(M)_\perp \qquad \text{and} \qquad {}^\perp F_H(e-p) = U_p(M)$$

and

$$U_p(M) \cong AB(F_H(p), B(H)_{sa}) \quad \textit{(Jordan isomorphism)}$$

where each weak-closed Jordan ideal* J *corresponds to the CP-split face* $F_H(c)$ *with central* $c \in \mathcal{U}(M)$, *and* $J = U_c(M) = c \circ M$.

PROOF: Clearly, $U_p(M)_\perp \subset \{p\}_\perp = F_H(e-p)$. Conversely, if $\psi \in F_H(e-p)$, $\psi = \psi_{e-p} = \psi \circ U_{e-p}$, so $\psi \circ (U_p(M)) = \psi((U_{e-p} \circ U_p)(M)) = 0$, i.e., $F_H(e-p) \subset U_p(M)_\perp$, hence $F_H(e-p) = U_p(M)_\perp$.

It is obvious that $F_H(e-p)^\perp \supset U_p(M)$. Conversely, $F_H(e-p)^\perp \subset F_{\mathbf{C}}(e-p)^\perp = U_p(M)$, so $F_H(e-p)^\perp = U_p(M)$.

For the isomorphism, consider the following diagram:

$$\begin{tikzcd}
U_p(M) \arrow[r, "i_p"] \arrow[dr, "k_p"'] & AB(F_H(p), B(H)_{sa}) \arrow[d, "j_p"] \\
 & A_0^b(F_{\mathbf{C}}(p), \mathbf{R})
\end{tikzcd}$$

where i_p and j_p are defined similarly as i and j in the proof of Theorem 3.7, and k_p denotes Alfsen-Shultz's duality ([5]). We can apply the same arguments as in the proof of Theorem 3.7 to the JW-algebra $U_p(M)$ to conclude that i_p is a Jordan isomorphism. ∎

We can generalize Theorem 4.10 as follows.

THEOREM 4.17. *Let* A *be a JC-algebra and* H *be a Hilbert space with* $dim H \geq \alpha_c(A)$. *Then, there exists one to one correspondence between the closed order ideal* I *of* A *and the BW-closed CP-projective face* $F_H(q)$ *such that*

$$F_H(q) = I^{\perp} \qquad \text{and} \qquad {}_{\perp}F_H(q) = I$$

If $p = e - q \in A$, then we have

$$I = U_p(A) \cong AC(F_H(p), B(H)_{sa})$$

In particular, each closed Jordan ideal J corresponds to the BW-closed CP-split face $F_H(q)$ with central idenpotent $q \in \mathcal{U}(A^{**})$, $p = e - q$, such that

$$J \cong AC(F_H(p), B(H)_{sa}) \qquad \text{and} \qquad A/J \cong AC(F_H(q), B(H)_{sa})$$

PROOF: Since $I \subset A$, $I^{\perp}$ is BW-closed CP-face, so that $I^{\perp} = F_H(q)$ for some closed $q \in \mathcal{U}(A^{**})$. Set $I' :=_{\perp} F_H(q)$, then I' is an order ideal of A, and clearly $I' \supset I$, hence we conclude $I = I'$ (using Hahn-Banach's theorem), so ${}_{\perp}F_H(q) = I$.

Let $p = e - q \in A$. By Theorem 4.16, $U_p(A^{**}) = {}^{\perp}F_H(q) \cong AB(F_H(p), B(H)_{sa})$. Since $p \in A$, by the similar arguments in Theorem 4.10, $U_p(A) = I \cong AC(F_H(p), B(H)_{sa})$.

If J is a closed Jordan ideal, J^{**} is a w*-closed Jordan ideal, so that J^{**} is of the form $U_p(A^{**})$ for some central $p \in \mathcal{U}(A^{**})$ (cf. [67; Lemma 2.1]). To show the isomorphisms, let $q = e - p$, then by the bipolar theorem and Theorem 4.16, ${}^{\perp}F_H(q) = J^{\circ\circ} = \sigma(A^{**}, A^{*}) - cl.\, J \cong J^{**} = U_p(A^{**}) \cong AB(F_H(p), B(H)_{sa})$. Hence we can identify $Q_H(J^{**})_n = Q_H(J) = F_H(p)$ and we have $J \cong AC(F_H(p), B(H)_{sa})$ where we consider that $F_H(p)$ has the BW-topology for J. Clearly, $A/J \cong AC(F_H(q), B(H)_{sa})$. ∎

Remark. In the above arguments, the basic concepts of Alfsen-Shultz's duality between unital JB-algebras [resp.JBW-algebras] and their state spaces [resp. normal state spaces] were generalized to a duality between ***non-unital*** JC-algebras [resp. JW-algebras] and their CP-state spaces [resp. normal CP-state spaces] with the following one to one correspondence,

$$p \in \mathcal{U}(M) \leftrightarrow \Phi_H(p)^\natural = U_p \in \mathcal{P}_H(M) \leftrightarrow F_H(p) = im\Phi_H(p) \cap Q_H(M)_n \in \mathcal{F}_H(M),$$

which is an order isomorphism as orthomodular lattice induced from $\mathcal{U}(M)$. We note again that, for C*- and W*-algebras, our CP-duality recovers the full C*-structure. We will see an advantage of this generalization in Chapter 7, where it will be applied to formulate simpler orientability conditions, which characterize C*-structures among Jordan structures.

CHAPTER 5

CP-MEASURE AND INTEGRATION

In this chapter, we develop the measure and integration theory inherent to CP-convexity, which will be applied in Chapter 6 to formulate the representation theory for CP-convex sets, i.e., a generalization of Choquet's theorem in the context of CP-convexity.

To illustrate our motivation, we recall that a scalar σ-convex combination is interpreted as the barycenter of an atomic probability measure. In analogy, a CP-convex combination in the CP-state space of a C*-algebra A,

$$\psi = \sum_{\alpha \in \Lambda} S_\alpha^* \psi_\alpha S_\alpha \quad \text{where} \quad \psi_\alpha \in Q_H(A) \quad \text{and} \quad S_\alpha \in B(H) \text{ with } \sum_{\alpha \in \Lambda} S_\alpha^* S_\alpha \leq I_H,$$

could be interpreted as a distribution of the weight of "effects" $S_\alpha^*(\cdot)S_\alpha$ in the sense of the theory of operations. Hence, in the scope of the CP-duality $A \cong AC(Q_H(A), B(H))$, we could claim that ψ be the barycenter of $Q_H(B(H))$-valued measure if the integral of the elements of $AC(Q_H(A), B(H))$ with respect to this measure is meaningful. We thus realize that we need to develop CP-map valued measure and integration theory. In the following, we actually formulate the theory for more generalized setting with $P(A, B)$-valued measure where $P(A, B)$ denotes the set of all positive operators between order unit spaces A and B, and define the intergral of A-valued strongly measurable functions with respect to this measure.

One will find that our integral is more straightforward than the usual method in vector measure theory in Banach spaces which uses the semi-variation (cf. Brooks-Dinculeanu ([20])), the latter being in the direction of a generalization of Bochner integral. Indeed, our integral can be regarded as a direct generaligation of (scalar) Lebesgue integral in its construction due to the order unit structure, which enables us to prove Lebesgue's dominated convergence theorem. We shall also discuss some aspects of the integral of weakly measurable functions. Since it seems hardly possible to formulate the general case for this weak integral, we focus mostly on the particular case where the $P(A, B)$-valued measure is "BW-absolutely continuous" with respect to a scalar measure, which, combined with the integral for strongly measurable functions, shall be enough for our purpose in Chapter 6.

Let $(X, \mathcal{B})$ be a measurable space, (A, e) be a (real) order unit space, and (B, e') be a (real) dual order unit space, i.e., there exists a base-norm space (B_*, K) such that $(B, e') = (B_*, K)^*$, where we denote the order units by e in A and e' in B respectively, and the base of B_* by K. (More generally, it shall be enough to assume that there exists an ordered real Banach space V in separating order and norm duality with B and that B is pointwise monotone σ-complete with respect to this duality.) We denote by $S(X, A)$ the set of all (countable valued) simple functions from X to A, i.e., $f \in S(X, A)$ is a function $f : X \to A$ of the form

$$f = \sum_{i=1}^{\infty} \chi_{E_i} a_i \quad \text{with} \quad (E_i)_{i=1}^{\infty} \subset \mathcal{B} \text{ (disjoint) and } \quad (a_i)_{i=1}^{\infty} \subset A,$$

where χ_E denotes the characteristic function of E. We denote by $BS(X, A)$ the set of all bounded elements in $S(X, A)$. It is straightforward to check that $BS(X, A)$

is a normed (real) linear space with the sup-norm; $\|f\| := \sup\{\|f(x)\|; x \in X\}$.

Let λ be a $P(A,B)$-valued BW-countably additive measure, i.e., $\lambda : X \to P(A,B)$ is a map such that, if $E = \sum_{i=1}^{\infty} E_i$ with $E_i \in \mathcal{B}$ (disjoint), then $\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i)$ converging in the BW-topology, i.e., for any $a \in A$ and $\rho \in B_*$, $(\lambda(E)a, \rho) = \sum_{i=1}^{\infty} (\lambda(E_i)a, \rho)$. We first claim the following lemma.

LEMMA 5.1. *Let $(E_i)_{i=1}^{\infty} \subset \mathcal{B}$ be disjoint and $(a_i)_{i=1}^{\infty} \subset A$ be a bounded sequence. Then the series $\sum_{i=1}^{\infty} \lambda(E_i)\, a_i$ converges in $\sigma(B, B_*)$-topology.*

PROOF: We first assume $(a_i) \subset A^+$ and set $b_n = \sum_{i=1}^{n} \lambda(E_i)\, a_i$ for $n \in \mathbb{N}$. Then b_n is an increasing sequence in B. We observe that

$$\begin{aligned} b_n &\leq \sum_{i=1}^{n} \lambda(E_i) \|a_i\| e \leq \Big(\sup_{1 \leq i \leq n} \|a_i\| \Big) \sum_{i=1}^{n} \lambda(E_i) e \\ &= \Big(\sup_{1 \leq i \leq n} \|a_i\| \Big)\, \lambda\Big(\sum_{i=1}^{n} E_i \Big) e \leq \Big(\sup_{i \in \mathbb{N}} \|a_i\| \Big)\, \lambda(X) e\,. \end{aligned}$$

From this, (b_n) is a bounded and increasing sequence in B, hence it converges to $\underset{n}{l.u.b.}(b_n) = \sum_{i=1}^{\infty} \lambda(E_i)\, a_i \in B$ in $\sigma(B, B_*)$-topology.

In case that $(a_i) \subset A$ is more general, we decompose $a_i = a_i^{(1)} - a_i^{(2)}$ with $a_i^{(1)}, a_i^{(2)} \in A^+$, and we define

$$\sum_{i=1}^{\infty} \lambda(E_i) a_i := \sum_{i=1}^{\infty} \lambda(E_i) a_i^{(1)} - \sum_{i=1}^{\infty} \lambda(E_i) a_i^{(2)}.$$

It is straightforward to check that this is well defined; indeed if $a_i = a_i^{(1)} - a_i^{(2)} = \tilde{a}_i^{(1)} - \tilde{a}_i^{(2)}$ where $a_i^{(j)}, \tilde{a}_i^{(j)} \in A^+$ $(j = 1, 2)$, then $a_i^{(1)} + \tilde{a}_i^{(2)} = \tilde{a}_i^{(1)} + a_i^{(2)}$, hence

$$\sum_{i=1}^{\infty} \lambda(E_i)(a_i^{(1)} + \tilde{a}_i^{(2)}) = \sum_{i=1}^{\infty} \lambda(E_i)(\tilde{a}_i^{(1)} + a_i^{(2)}),$$

i.e.,

$$\sum_{i=1}^{\infty} \lambda(E_i)a_i^{(1)} + \sum_{i=1}^{\infty} \lambda(E_i)\tilde{a}_i^{(2)} = \sum_{i=1}^{\infty} \lambda(E_i)\tilde{a}_i^{(1)} + \sum_{i=1}^{\infty} \lambda(E_i)a_i^{(2)},$$

so we have

$$\sum_{i=1}^{\infty} \lambda(E_i)a_i^{(1)} - \sum_{i=1}^{\infty} \lambda(E_i)a_i^{(2)} = \sum_{i=1}^{\infty} \lambda(E_i)\tilde{a}_i^{(1)} - \sum_{i=1}^{\infty} \lambda(E_i)\tilde{a}_i^{(2)}.$$

∎

Remark. The above lemma holds if λ is finitely additive at e.

We are now ready to define the integral of bounded simple functions.

DEFINITION 5.2. *Let $\lambda : \mathcal{B} \to P(A, B)$ be a BW-countably additive measure and let $f \in BS(X, A)$ which is expressed as*

$$f = \sum_{i=1}^{\infty} \chi_{E_i} a_i \quad \text{with } (E_i) \subset \mathcal{B} \text{ (disjoint) and } (a_i) \subset A \text{ (bounded).}$$

*We define the **integral** of f with respect to λ by*

$$\int f \, d\lambda := \sum_{i=1}^{\infty} \lambda(E_i)a_i \quad \in B,$$

which converges in $\sigma(B, B_*)$-topology by Lemma 5.1.

Remark. The above integral is well defined, i.e., it does not depend on any particular way of the expression of $f \in BS(X, A)$. It suffices to prove this for $f \in BS(X, A)^+$ from the definition of the integral. Suppose that f has two different expressions

$$f = \sum_{i=1}^{\infty} \chi_{E_i} a_i = \sum_{j=1}^{\infty} \chi_{F_j} b_j \quad \text{where } X = \cup_{i=1}^{\infty} E_i = \cup_{j=1}^{\infty} F_j \text{ (disjoint)}$$

$$\text{and} \quad (a_i), (b_j) \subset A^+ \quad \text{(bounded).}$$

We set $G_{ij} := E_i \cap F_j$ and $c_{ij} := a_i = b_j$ (on G_{ij}), where we can assume without loss of generality that $c_{ij} \neq 0$. Then, $E_i = \cup_{j=1}^{\infty} G_{ij}$ (disjoint) and $F_j = \cup_{i=1}^{\infty} G_{ij}$ (disjoint), so that, for any $\rho \in (B_*)^+$, we have

$$\begin{aligned}(\sum_{i=1}^{\infty} \lambda(E_i)a_i, \rho) &= \sum_{i=1}^{\infty}(\lambda(\cup_{j=1}^{\infty} G_{ij})a_i, \rho) = \sum_{i=1}^{\infty}(\sum_{j=1}^{\infty}(\lambda(G_{ij})c_{ij}), \rho) \\ &= \sum_{j=1}^{\infty}(\sum_{i=1}^{\infty}(\lambda(G_{ij})c_{ij}, \rho)) = \sum_{j=1}^{\infty}(\lambda(\cup_{i=1}^{\infty} G_{ij})b_j, \rho) \\ &= (\sum_{j=1}^{\infty} \lambda(F_j)b_j, \rho),\end{aligned}$$

noting that λ is BW-countably additive, and the sums $\sum_{i=1}^{\infty}$ and $\sum_{j=1}^{\infty}$ are exchangable since all terms are positive. Hence, with the duality $B \cong A^b(N(B))$, we have

$$\sum_{i=1}^{\infty} \lambda(E_i)a_i = \sum_{j=1}^{\infty} \lambda(F_j)b_j.$$

The proof of the following lemma is immediate from the definition. This will be essential for defining the integral for general strongly measurable functions.

LEMMA 5.3. *The integral $\lambda(f) := \int f \, d\lambda$ for $f \in BS(X, A)$ defined in Definition 5.2 is a bounded linear map from $BS(X, A)$ to B with*

$$\|\lambda(f)\| = \|\int f \, d\lambda\| \leq \|\lambda\|\|f\| \quad \textit{where } \|\lambda\| = \|\lambda(X)e\|.$$

We recall that a function $f : X \to A$ is said to be ***strongly measurable*** if its range $f(X)$ is norm separable in A, and $f^{-1}(S_r(a)) \in \mathcal{B}$ for all $a \in A$ and $r > 0$ where $S_r(a) = \{b \in A; \|b - a\| < r\}$. We denote by $M_s(X, A)$ [resp. $BM_s(X, A)$] the set of all strongly measurable [resp. bounded strongly measurable] functions from X to A. The following fact is well known (see e.g., [27; II. Cor.3]), however we state the proof here briefly for convenience of the reader.

LEMMA 5.4. *$BS(X,A)$ is dense in $BM_s(X,A)$ with the uniformly convergence topology.*

PROOF: Let $f \in BM_s(X,A)$, and let $(a_k)_{k=1}^{\infty}$ be a norm dense subset of $f(X)$. For each $n \in \mathbb{N}$, we define $Y_k^{(n)} := f^{-1}(S_{\frac{1}{n}}(a_k) \cap f(X)) \in \mathcal{B}$ and set $X_k^{(n)} := Y_k^{(n)} \setminus \cup_{l<k} Y_l^{(n)} \in \mathcal{B}$. Then, for each fixed $n \in \mathbb{N}$, $X_k^{(n)}$ are mutually disjoint and $X = \cup_{k=1}^{\infty} X_k^{(n)}$. We then define $f_n := \sum_{k=1}^{\infty} \chi_{X_k^{(n)}} a_k \in BS(X,A)$. f_n converges uniformly to f, since for every $n \in \mathbb{N}$ and $x \in X$, there exists $k \in \mathbb{N}$ such that $x \in X_k^{(n)}$ and we have $\|f_n(x) - f(x)\| = \|a_k - f(x)\| < \frac{1}{n}$. ∎

DEFINITION 5.5. *From Lemmas 5.3 and 5.4, the map $\lambda : BS(X,A) \to B$ has the unique bounded linear extension $\bar{\lambda}$ to $BM_s(X,A)$. We define this map the* integration *of elements in $BM_s(X,A)$ with respect to λ, and denote its image by $\int f \, d\lambda$, i.e.,*

$$\begin{tikzcd}
\lambda : \arrow[d] & BS(X,A) \arrow[r] \arrow[d, phantom, "\cap"] & B \\
\bar{\lambda} : & BM_s(X,A) \arrow[ur] &
\end{tikzcd}$$

(In the following, we will use the same λ for this extension $\bar{\lambda}$.)

We then define the integral for unbounded strongly measurable functions as follows. Let $f \in M_s(X,A^+)$ and set $X_N := \{x \in X; \|f(x)\| \le N\}$. Then, the sequence $(\int_{X_N} f \, d\lambda)_{N=1}^{\infty}$ is increasing, so if it is bounded, then it converges, and we say f is integrable *over X with respect to λ and define*

$$\int_X f \, d\lambda := \lim_{N \to \infty} \int_{X_N} f \, d\lambda.$$

If A is a complex vector space whose self-adjoint part is an order unit space, then for $f \in M_s(X, A)$, we decompose $f = f^{(1)} - f^{(2)} + i(f^{(3)} - f^{(4)})$ where $f^{(k)} \in M_s(X, A^+)$ for $1 \leq k \leq 4$, and we say that f is **integrable** *with respect to λ if all $f^{(k)}$ are integrable, and define*

$$\int_X f\, d\lambda := \int_X f^{(1)}\, d\lambda - \int_X f^{(2)}\, d\lambda + i(\int_X f^{(3)}\, d\lambda - \int_X f^{(4)}\, d\lambda).$$

We denote by $L^1_\lambda(X, A)$ the set of all integrable functions in $M_s(X, A)$ with respect to λ.

Remarks. 1. It is obvious that the above integral is linear with respect to $f \in BM_s(X, A)$. We will prove the linearity of the integral for general $f \in L^1_\lambda(X, A)$ later in Corollary 5.10.

2. We define λ-null set E by $\lambda(E) = 0$ (i.e., $\lambda(E)e = 0$). We can define λ- a.e. as in the usual Lebesgue integral, and we can consider that $L^1_\lambda(X, A)$ is the set of equivalence classes with $f \sim g$ iff $f = g$ λ- a.e..

In the remainder of this chapter, we assume that A is a real order unit space, noting that the generalization to the complex case is straightforward as above.

We immediately have the uniform convergence theorem.

PROPOSITION 5.6. *Let $(f_n)_{n=1}^\infty \in L^1_\lambda(X, A)$ be uniformly bounded (hence $\subset BM_s\ (X, A)$) and $f_n \to f$ uniformly λ- a.e. on X, then $\int_X f_n\, d\lambda \to \int_X f\, d\lambda$ in the norm in B.*

PROOF: Note the following inequality;

$$\|\int_X f_n \, d\lambda - \int_X f \, d\lambda\| = \|\int_X (f_n - f) \, d\lambda\| \leq \int_X \|f_n - f\|e \, d\lambda$$
$$\leq \text{ess. } \sup_{x \in X} \|f_n(x) - f(x)\| \; \|\lambda(X)e\|.$$

∎

The following proposition shows that the sequence in Lemma 5.4 can be chosen so that it is monotone increasing if A is separable.

PROPOSITION 5.7. *Let A be a separable order unit space and let $f \in M_s$ (X, A^+). Then, there exists a sequence $(f_n)_{n=1}^{\infty} \subset BS(X, A^+)$ such that $(f_n)_{n=1}^{\infty}$ is monotone increasing and converges to f point wise. If $f \in BM_s(X, A^+)$, then (f_n) can be chosen such that $f_n \nearrow f$ uniformly.*

PROOF: We first consider the case $f \in BM_s(X, A^+)$. We can assume without loss of generality that $\|f(x)\| \leq 1$ for all $x \in X$. Let $(a_k)_{k=0}^{\infty}$ be a dense sequence in $A_1^+ = \{a \in A^+; \|a\| \leq 1\}$ where we set $a_0 = 0$, and let $(\epsilon_n)_{n=1}^{\infty}(0 < \epsilon_n < 1)$ be a monotone decreasing sequence which converges to 0, and we take $(b_k^{(1)})_{k=1}^{\infty} \subset A^+$ such that $b_k^{(1)} \in S_{\epsilon_1}(a_k)$ and $0 \leq b_k^{(1)} \leq a_k$ $(b_0^{(1)} = 0)$. We define

$$V_k^{(1)} := \{a \in A; a \in S_{\epsilon_1}(a_k) \text{ and } a \geq b_k^{(1)}\}.$$

We first claim that $f(X) \subset \cup_{k=1}^{\infty} V_k^{(1)}$. In fact, for $x \in X$ such that $f(x) \neq 0$, let

$$W_x^{(1)} := S_{\epsilon_1}(f(x)) \cap (f(x) - (A^+)^{\circ}).$$

Since $(a_k)_{k=1}^{\infty}$ is dense in A_1^+, there exists some $a_{k_0} \in W_x^{(1)}$ which is not empty since $f(x) \neq 0$. If $f(x) = 0$, we set $a_{k_0} = 0$. It follows that $f(x) \in S_{\epsilon_1}(a_{k_0})$

and $f(x) \geq a_{k_0} \geq b_{k_0}^{(1)}$, i.e., $f(x) \in V_{k_0}^{(1)}$, which proves our claim. If we define $U_k^{(1)} := V_k^{(1)} \setminus \cup_{l<k} V_l^{(1)}$, which are mutually disjoint, then $f(X) \subset \cup_{k=1}^{\infty} U_k^{(1)}$ (disjoint sum).

As the next step, for each $l \in \mathbf{N}$, and $a_k \in V_l^{(1)}$, we take $b_{l,k}^{(2)} \in A^+$ such that $b_{l,k}^{(2)} \in S_{\epsilon_2}(a_k) \cap V_l^{(1)}$ and $b_l^{(1)} \leq b_{l,k}^{(2)} \leq a_k$, and we define

$$V_{l,k}^{(2)} := \{a \in A; a \in S_{\epsilon_2}(a_k) \text{ and } a \geq b_{l,k}^{(2)}\} \cap U_l^{(1)}.$$

By the same arguments as above, we can show $f(X) \cap U_l^{(1)} \subset \cup_{k=1}^{\infty} V_{l,k}^{(2)}$, hence, by reordering $V_{l,k}^{(2)}$ $(l, k = 1, 2, \cdots)$ and denoting them by $V_m^{(2)} (m \in \mathbf{N})$, we have $f(X) \subset \cup_{m=1}^{\infty} V_m^{(2)}$. Defining $U_m^{(2)} := V_m^{(2)} \setminus \cup_{l<m} V_l^{(2)}$, we get $f(X) \subset \cup_{m=1}^{\infty} U_m^{(2)}$ (disjoint sum).

We continue the same procedure to deduce $f(X) \subset \cup_{m=1}^{\infty} U_m^{(n)}$ (disjoint sum), where for each $n \in \mathbf{N}$, $U_m^{(n)}$ are mutually disjoint Borel set induced from the norm topology in A. For each $n \in \mathbf{N}$, we define $X_m^{(n)} := f^{-1}(U_m^{(n)} \cap f(X)) \in \mathcal{B}$, then $X = \cup_{m=1}^{\infty} X_m^{(n)}$ (disjoint sum), and we set

$$f_n := \sum_{m=1}^{\infty} \chi_{X_m^{(n)}} b_m^{(n)} \in BS(X, A),$$

where $b_m^{(n)}$ denotes $b_{l,k}^{(n)}$ which was used to define $V_m^{(n)} = V_{l,k}^{(n)}$.

It follows from the construction above that $f_n(x)$ is increasing for each $x \in X$, since if $f(x) \in U_m^{(n)}$, which is contained in some $U_{m'}^{(n-1)}$, so $b_m^{(n)} \geq b_{m'}^{(n-1)}$. It is also straightforward to see that the convergence is uniform for $x \in X$, since for each $n \in \mathbf{N}$ and $x \in X$, there exists $m \in \mathbf{N}$ such that $x \in X_m^{(n)}$, i.e., $f(x) \in U_m^{(n)}$, then observe that

$$\|f(x) - f_n(x)\| = \|f(x) - b_m^{(n)}\| = \|f(x) - b_{l,k}^{(n)}\| \leq \|f(x) - a_k\| + \|a_k - b_{l,k}^{(n)}\| < 2\epsilon_n.$$

It is left to prove the general case that f is unbounded, i.e., $f \in M_s(X, A^+)$. We set $Y_n := \{x \in X;\ n-1 < \|f(x)\| \leq n\}$ and define $f^{(n)} := \chi_{Y_n} f \in BM_s(X, A^+)$. From our first step just proved, we can take a sequence $(f_k^{(n)})_{k=1}^{\infty} \subset BS(X, A^+)$ such that $f_k^{(n)} \nearrow f^{(n)}$ $(k \to \infty)$. Set $f_n := \sum_{m=1}^{n} f_n^{(m)} \in BS(X, A^+)$, then $f_n \nearrow f$ (point wise). This completes the proof. ▮

Hereafter, A is not required to be separable. To prove the monotone convergence theorem and Lebesgue's dominated convergence theorem, we need some preparations.

PROPOSITION 5.8. *Let $(f_i)_{i=1}^{\infty}$ and $(g_j)_{j=1}^{\infty}$ be increasing sequences in BM_s (X, A^+) and suppose that $\lim_{i\to\infty} f_i = \lim_{j\to\infty} g_j \in M_s(X, A^+)$ for λ-* a.e. *on X. Then*

$$\lim_{i\to\infty} \int_X f_i \, d\lambda = \lim_{j\to\infty} \int_X g_j \, d\lambda.$$

PROOF: We can assume that $f_i(x)$ and $g_j(x)$ $(i, j = 1, 2 \cdots)$ and $h(x) = \lim_{i\to\infty} f_i(x) = \lim_{j\to\infty} g_j(x)$ are everywhere defined by neglecting λ-null set.

Let $j_0 \in J$ be fixed. We first claim that

$$\lim_{i\to\infty} \int_X f_i \, d\lambda \geq \int_X g_{j_0} \, d\lambda.$$

To show this, for any fixed $\epsilon > 0$, we define

$$F_i := \{x \in X; f_i(x) \geq g_{j_0}(x) - \epsilon e\}.$$

It is clear that $F_i \nearrow X$, since $\lim_{i\to\infty} f_i(x) \geq g_{j_0}(x) > g_{j_0}(x) - \epsilon e$ and note here that $f_n(x)$ converges to $f(x)$ for each $x \in X$ in the order unit norm of A, which is a direct consequence of Dini's theorem applied to $A \cong AC(Q(A), \mathbf{R})$. Hence it follows from the BW-countably additivity of λ that $\lim_{i\to\infty} \lambda(X \setminus F_i) \to 0$ in BW-topology, i.e., for the given $\epsilon > 0$, and for an arbitrary $\rho \in B_*^+$, there exists $i_0 \in \mathbf{N}$ such that if $i \geq i_0$ then we have $0 < (\lambda(X \setminus F_i)e, \rho) < \epsilon$. Observe that

$$\begin{aligned}\int_X f_i \, d\lambda \geq \int_{F_i} f_i \, d\lambda &\geq \int_{F_i} (g_{j_0}(x) - \epsilon e) \, d\lambda \\ &= \int_X g_{j_0}(x) \, d\lambda - \int_{X\setminus F_i} g_{j_0}(x) \, d\lambda - \epsilon \int_{F_i} e \, d\lambda \\ &\geq \int_X g_{j_0}(x) \, d\lambda - \|g_{j_0}\| \lambda(X \setminus F_i)e - \epsilon\lambda(X)e.\end{aligned}$$

So, for any $\rho \in B_*^+$ and $i \geq i_0$,

$$\left(\int_X f_i \, d\lambda - \int_X g_{j_0} \, d\lambda, \rho\right) \geq -\epsilon(\|g_{j_0}\| + \|\lambda\|).$$

Since $\epsilon > 0$ is arbitrary, we have shown

$$\lim_{i\to\infty} \int_X f_i \, d\lambda \geq \int_X g_{j_0} \, d\lambda,$$

which was our claim in the beginning of the proof.

We next let $j_0 \to \infty$, and get

$$\lim_{i\to\infty} \int_X f_i \, d\lambda \geq \lim_{j\to\infty} \int_X g_j \, d\lambda.$$

By symmetry, we have

$$\lim_{i\to\infty} \int_X f_i \, d\lambda \leq \lim_{j\to\infty} \int_X g_j \, d\lambda,$$

hence we have proved the proposition. ∎

The monotone convergence theorem is an immediate consequence of the above proposition.

COROLLARY 5.9. *Let $(f_n)_{n=1}^{\infty} \subset BM_s(X, A)$ be a monotone increasing sequence which converges to $f \in M_s(X, A)$ (λ- a.e.), then*

$$\lim_{n \to \infty} \int f_n \, d\lambda = \int f \, d\lambda$$

PROOF: We can assume $f \geq 0$. If $f \in BM_s(M, A^+)$, set $g_j = f$ for all $j \in \mathbb{N}$ in Proposition 5.8. If f is unbounded, set $g_j = \chi_{X_j} f$ with $X_j = \{x \in X; \|f(x)\| \leq j\}$ in Proposition 5.8. ∎

Remark. From this corollary, if A is a separable order unit space, we could define the integral of $f \in M_s(X, A^+)$ by the limit $\lim\limits_{n \to \infty} \int f_n \, d\lambda$ where $(f_n) \subset BS(X, A^+)$ is the increasing sequence of Proposition 5.7 if the limit exists.

COROLLARY 5.10. *The integral in Definition 5.5 is linear, i.e., for any $f, g \in L^1_\lambda(X, A)$ and $c \in \mathbb{C}$,*

$$\int_X (f+g) \, d\lambda = \int_X f \, d\lambda + \int_X g \, d\lambda \qquad \text{and} \qquad \int_X cf \, d\lambda = c \int_X f \, d\lambda.$$

PROOF: We first assume $f, g \in L^1_\lambda(X, A^+)$. Let $X_N = \{x \in X; \|f(x) + g(x)\| \leq N\}$. Then, it is clear that $\chi_{X_N} f, \chi_{X_N} g \in BM_s(X, A)$ and $\chi_{X_N} f \nearrow f, \chi_{X_N} g \nearrow g$ as $N \to \infty$, so that by Corollary 5.9, $\int_X f \, d\lambda = \lim_{N \to \infty} \int_{X_N} f \, d\lambda$ and $\int_X g \, d\lambda = \lim_{N \to \infty} \int_{X_N} g \, d\lambda$, from which we have

$$\begin{aligned}
\int_X (f+g) \, d\lambda &= \lim_{N \to \infty} \int_{X_N} (f+g) \, d\lambda \\
&= \lim_{N \to \infty} \Big(\int_{X_N} f \, d\lambda + \int_{X_N} g \, d\lambda \Big) \\
&= \int_X f \, d\lambda + \int_X g \, d\lambda.
\end{aligned}$$

Since $\int_X cf\,d\lambda = c\int_X f\,d\lambda$ for $c > 0$ is obvious, the integral is an affine map from $L^1_\lambda(X, A)^+$ to B^+. The rest of the proof is completed under the same argument in convexity theory that every affine map on a positive cone which vanishes at 0 can be extended to a linear map on the complex linear space which is generated by the cone. ∎

Our next concern should be about Lebesgue's dominated convergence theorem. We approach this by generalizing Egoroff's theorem.

THEOREM 5.11. *Let* $(f_n)_{n=1}^{\infty} \subset M_s(X, A)$ *be a sequence such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *in the norm of* A *for* λ- a.e. $x \in X$. *Then, for any* $\epsilon > 0$, *and* $\rho \in B_*$, *there exists* $Y \in \mathcal{B}$ *with* $|(\lambda(X \setminus Y)e, \rho)| < \epsilon$ *such that* (f_n) *converges to* f *uniformly on* Y.

PROOF: We can assume $\|f_n(x) - f(x)\| \to 0$ for all $x \in X$ negrecting λ- null set. For each $n, p \in \mathbb{N}$, we set

$$X_{n,p} = \bigcap_{k\geq n} \{x \in X; \|f_k(x) - f(x)\| < \frac{1}{2^p}\} \in \mathcal{B}.$$

Then, for each fixed $p \in \mathbb{N}$, $X_{n,p} \nearrow X$ (as $n \to \infty$), so $\lambda(X_{n,p}) \longrightarrow \lambda(X)$ $(n \to \infty)$ in BW-topology. It follows that for any $\epsilon > 0$ and $\rho \in B_*$, there exists $N_p \in \mathbb{N}$ such that, if $n \geq N_p$, then $|(\lambda(X \setminus X_{n,p})e, \rho)| < \epsilon/2^p$. Set $X_p := X_{N_p,p}$ and $Y := \cap_{p=1}^{\infty} X_p \in \mathcal{B}$. Then, for any $\rho \in B_*$,

$$|(\lambda(X \setminus Y)e, \rho)| = |\lambda(\cup_{p=1}^{\infty}(X \setminus X_p)e, \rho)| \leq \sum_{p=1}^{\infty} |(\lambda(X \setminus X_p)e, \rho)| < \sum_{p=1}^{\infty} \frac{\epsilon}{2^p} = \epsilon.$$

Note also that, for any $p \in \mathbb{N}$, if $k \geq N_p$, then $\|f_k(x) - f(x)\| < 1/2^p$ for all $x \in Y (\subset X_p)$, which implies that (f_n) converges to f uniformly. ∎

The following proposition shows that the integral is absolutely continuous with respect to BW-topology.

PROPOSITION 5.12. *Let* $f \in L^1_\lambda(X, A)$. *Then, for any* $\epsilon > 0$, *and* $\rho \in B_*$, *there exists* $\delta > 0$ *such that, if* $(\lambda(E)e, \rho) < \delta$ *for* $E \in \mathcal{B}$, *then* $|(\int_E f\, d\lambda, \rho)| < \epsilon$.

PROOF: We first assume $f \geq 0$. From Definition 5.5, for $X_n = \{x \in X; \|f(x)\| \leq n\}$, we have $\lim_{n\to\infty} \int_{X_n} f\, d\lambda = \int f\, d\lambda$, i.e., for $\epsilon > 0$ and for a large $n \in \mathbb{N}$, $|(\int_{X\setminus X_n} f\, d\lambda, \rho)| < \epsilon/4$. We set $\delta = \epsilon/4n$. If $(\lambda(E)e, \rho) < \delta$ for $E \in \mathcal{B}$, then

$$|(\int_{E\cap X_n} f\, d\lambda, \rho)| \leq n(\lambda(E\cap X_n)e, \rho) \leq n\delta < \epsilon/4$$

and

$$|(\int_{E\setminus X_n} f\, d\lambda, \rho)| \leq |(\int_{X\setminus X_n} f\, d\lambda, \rho)| < \epsilon/4,$$

hence

$$|(\int_E f\, d\lambda, \rho)| \leq |(\int_{E\cap X_n} f\, d\lambda, \rho)| + |(\int_{E\setminus X_n} f\, d\lambda, \rho)| < \epsilon/2 < \epsilon.$$

For the general case, we decompose $f = f^{(1)} - f^{(2)}$ with $f^{(1)}, f^{(2)} \in L^1_\lambda(X, A^+)$, and assume that $\delta^{(1)}, \delta^{(2)}$ are chosen as above. Set $\delta = \min\{\delta^{(1)}, \delta^{(2)}\}$. Then, if $(\lambda(E)e, \rho) < \delta$, we have

$$|(\int_E f\, d\lambda, \rho)| = |(\int_E f^{(1)}\, d\lambda, \rho)| + |(\int_E f^{(2)}\, d\lambda, \rho)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This completes the proof. ∎

Now, Lebesgue's dominated convergence theorem is generalized as follows.

THEOREM 5.13. *Let $(f_n)_{n=1}^{\infty} \subset M_s(X, A)$ be a sequence such that $\lim_{n\to\infty} f_n(x) = f(x)$ for λ- a.e. $x \in X$, and suppose that it is order bounded by $g \in L^1_\lambda(X, A^+)$, i.e., $-g(x) \leq f_n(x) \leq g(x)$ for λ- a.e. $x \in X$ and all $n \in \mathbb{N}$. Then, $f \in L^1_\lambda(X, A)$ and*

$$\lim_{n\to\infty} \int_X f_n \, d\lambda = \int_X f \, d\lambda$$

in $\sigma(B, B_*)$-topology.

PROOF: Since the order interval $[-g(x), g(x)]$ is norm closed, $-g(x) \leq f(x)$ $(= \lim_{n\to\infty} f_n(x)) \leq g(x)$ for λ- a.e. $x \in X$, hence $\| \int_X f \, d\lambda\| \leq \| \int_X g \, d\lambda\| < \infty$, i.e., $f \in L^1_\lambda(X, A)$.

From Proposition 5.12, for any ϵ, and $\rho \in B_*$, there exists $\delta > 0$ such that, if $(\lambda(E)e, \rho) < \delta$, then $|(\int_E g \, d\lambda, \rho)| < \epsilon$. By Theorem 5.11, for this chosen $\delta > 0$, and for the same $\rho \in B_*$, there exists $Y \in \mathcal{B}$ with $|(\lambda(X \setminus Y)e, \rho)| < \delta$ such that f_n converges to f uniformly on Y, i.e., for large n, $\|f_n(y) - f(y)\| < \epsilon$ for all $y \in Y$. We then have

$$\begin{aligned} |(\int_X (f_n - f) \, d\lambda, \rho)| &\leq |(\int_Y (f_n - f) \, d\lambda, \rho)| + |(\int_{X\setminus Y} (f_n - f) \, d\lambda, \rho)| \\ &\leq \epsilon(\lambda(Y)e, \rho) + 2|(\int_{X\setminus Y} g \, d\lambda, \rho)| \leq \epsilon(\|\lambda\| + 2) \end{aligned}$$

This shows $\lim_{n\to\infty} f_n \, d\lambda = \int f \, d\lambda$ in $\sigma(B, B_*)$-topology. ∎

Remark. Brooks-Dinculeanu [20] proved in the general setting for Banach spaces that the Lebesgue's dominated convergence theorem holds in the sense of convergence in measure (cf.[20; Th.2.6 and §4]). The above result shows that, in the setting of order unit spaces, we have Lebesgue's theorem in a stronger sense, namely in the sense of a.e.- convergence.

We next proceed to investigate the integral of weakly measurable functions. After some attempts, one realizes that it is truly difficult to define the integral of weakly measurable functions in the strong sense, i.e., using approximation by simple functions. However, recently there have been notable progress about weakly measurable functions taking the values in a locally convex Suslin space and their integrals with respect to a scalar measure (see [18], [77]). This restriction does not affect our purpose where we can assume that the functions are taking values in a w*-separable dual order unit space, hence a Suslin space. We will show that we can define an integral of w*-measurable functions in the strong sense with respect to a positive operator map valued measure under some constraint. (Recall that a Hausdorff topological space is defined to be a Suslin space if it is a continuous image of a Polish space.)

In what follows, we assume that A and B are dual order unit spaces and A is w*-separable, and $\lambda : \mathcal{B} \to P(A,B)_n$ is a BW-countably additive measure where $P(A,B)_n$ denotes the set of all normal (i.e., $\sigma(A,A_*)-\sigma(B,B_*)$ continuous) positive linear maps from A to B. We denote by $M_w(X,A)$ [resp. $BM_w(X,A)$] the set of all w*- (i.e.,$\sigma(A,A_*)$)-measurable [resp. bounded w*-measurable] functions from X to A.

From Blondia's result ([18; Prop.2.3]), for any $f \in BM_w(X,A)$, there exists a sequence $(f_n(x))$ of finite valued simple functions which converges to $f(x)$ in $\sigma(A,A_*)$-topology for each $x \in X$. Let

$$f_n(x) = \sum_{i=1}^{m_n} \chi_{E_i}^{(n)} a_i^{(n)} \quad \text{with } (E_i^{(n)}) \subset \mathcal{B} \text{ (diajoint)} \quad \text{and} \quad (a_i^{(n)}) \subset A$$

and define

$$b_n := \int_X f_n \, d\lambda = \sum_{i=1}^{m_n} \lambda(E_i^{(n)}) a_i^{(n)} \in B,$$

which is a particular case in Definition 5.2.

Since $\|b_n\| = \| \int_X f_n \, d\lambda\| \leq \|\lambda\| M$ where $M = \sup_n \|f_n\|$, $(b_n)_n$ is bounded, so is relatively $\sigma(A, A_*)$-compact, hence it has at least one accumulation point in B. If it is unique, we could define the integral $\int_X f \, d\lambda$ by its limit. In the following, we will give a sufficient condition for this purpose.

DEFINITION 5.14. *Let $\lambda : \mathcal{B} \to P(A, B)$ be a BW-countably additive measure. For each $a \in A$, $\rho \in B_*$, and $E \in \mathcal{B}$, define $\lambda_{a,\rho}(E) := (\lambda(E)a, \rho)$. If there exists a scalar measure μ such that $\lambda_{a,\rho} \ll \mu$ uniformly for $a \in A$ and $\rho \in B_*$. then we say that λ is **BW-absolutely continuous with respect to** μ.*

PROPOSITION 5.15. *Let A and B be dual order unit spaces and assume that A is w*-separable, and let $\lambda : \mathcal{B} \to P(A, B)_n$ be a normal BW-countably additive measure. If λ is BW-absolutely continuous with respect to a scalar measure. then $(b_n) = (\int_X f_n \, d\lambda)$ is a Cauchy sequence in $\sigma(B, B_*)$-topology, hence has a unique limit in B.*

PROOF: Since $\lambda_{a,\rho} \ll \mu$, where μ is a scalar measure, by Lebesgue-Radon-Nikodym's theorem, there exists $\varphi(a, \rho) \in L^1_\mu$ such that

$$\lambda_{a,\rho}(E) = (\lambda(E)a, \rho) = \int_E \varphi(a, \rho)(\zeta) \, d\mu(\zeta).$$

We note that, from the above form, for any fixed $\rho \in B_*$ and $E \in \mathcal{B}$, $\int_E \varphi(a, \rho)(\zeta) \, d\mu$ is $\sigma(A, A_*)$-continuous.

Observe that, for each $\rho \in B_*$,

$$\begin{aligned}(b_m - b_n, \rho) &= (\int_X (f_m - f_n)\, d\lambda, \rho) \\ &= \sum_{i,j} (\lambda(E_i^{(m)} \cap E_j^{(n)})(a_i^{(m)} - a_j^{(n)}), \rho) \\ &= \sum_{i,j} \int_{E_i \cap E_j} \varphi(a_i^{(m)} - a_j^{(n)}, \rho)\, d\mu(\zeta) \\ &= \int_X \varphi(f_m(\zeta) - f_n(\zeta), \rho)\, d\mu(\zeta)\end{aligned}$$

Since $(f_m(\zeta) - f_n(\zeta)) \to 0$ as $m, n \to \infty$ in $\sigma(A, A_*)$-topology for μ-a.e. $\zeta \in X$, from the above remark we conclude $\int_X \varphi(f_m(\zeta) - f_n(\zeta), \rho)\, d\mu(\zeta) \to 0$ as $m, n \to \infty$, hence $(b_m - b_n) \to 0$ (as $m, n \to \infty$) in $\sigma(B, B_*)$-topology. This completes the proof. ∎

Remarks. *1.* By the linear dependence on ρ in $\varphi(a, \rho)(\zeta)$, we can write $\varphi(a, \rho)(\zeta) = (\varphi(a)(\zeta), \rho)$, so that $(b, \rho) = \int_X (\varphi(f(\zeta)), \rho)\, d\mu(\zeta)$. This shows that $b = \int_X \varphi(f(\zeta))\, d\mu(\zeta)$ (weak*-integral).

2. At this moment, we do not have the integral of weakly measurable functions in the general setting. We could take the alterative way using monotone converging simple functions if we have the weak version of Proposition 5.7. However, our proof of Proposition 5.7 cannot be applied for this case since the interior $(A^+)^\circ$ in w*-topology is not assured to be non-empty in general. In fact, if A^+ has non-empty interior in w*-topology, then every positive linear form is w*-continuous (cf. [66; Th. 5.5]), which implies that A is reflexive, hence essentially reduced to the finite dimensional case.

3. In Chapter 6, we will apply our integral to the situation where λ is a CP-map valued measure, which we call "CP-measure and integral". More precicely,

letting $\mathcal{A}$ be a C*-algebra and H be a Hilbert space, we will set

$X = Q_H(\mathcal{A})$

$\mathcal{B} = \{$the Borel sets induced from the norm topology or the BW-topology on $Q_H(\mathcal{A})\ \}$

$A = B = B(H) = (\mathcal{T}(H))^*$

$\lambda : \mathcal{B} \to Q_H(B(H))$ or $Q_H(B(H))_n$ (BW-countably additive measure)

$f \in BM_s(Q_H(\mathcal{A}), B(H))$ or $BM_w(Q_H(\mathcal{A}), B(H))$

If H is separable, then A is w*-separable, hence satisfies the requirement of Proposition 5.15.

CHAPTER 6

CP-DECOMPOSITION AND CP-CHOQUET THEOREM

It is a fundamental problem in mathematics and its applications to decompose a mathematical object into "pure" phases. In this chapter, we consider the decomposition problem for completely positive maps and we apply this result to generalize Choquet's theorem for CP-state spaces in the context of CP-convexity.

We first note that the pure phase decomposition for CP-maps has a difficulty in its nature. To illustrate this fact, let A be a C*-algebra, H be a Hilbert space and let $\psi = V^*\pi V \in CP(A, B(H))$, and suppose that π has a irreducible decomposition, i.e.,

$$\pi = \int_Z^{\oplus} \pi(\zeta)\, d\mu(\zeta) \qquad \text{and} \qquad H_\pi = \int_Z^{\oplus} H_\pi(\zeta)\, d\mu(\zeta).$$

Corresponding to this, it would be quite natural to claim that ψ could be decomposed into pure CP-maps of the following form

$$\psi = \int_Z^{\oplus} V(\zeta)^*\pi(\zeta)V(\zeta)\, d\mu(\zeta),$$

where $V(\zeta)$ could be defined as

$$V(\zeta) : h \mapsto Vh(\zeta) \quad \text{for } h \in H.$$

Unfortunately, one will realize here that $V(\zeta)$ is not bounded in general, hence the integrand $V(\zeta)\pi(\zeta)V(\zeta)$ is not a pure CP-map; one can see this in the particular case where $V = id$ $(H = H_\pi)$, i.e., $V(\zeta) : h \in H_\pi \mapsto h(\zeta) \in H_\pi(\zeta)$ and $\|h\|^2 = \int_Z^\oplus \|h(\zeta)\|^2 \, d\mu(\zeta)$, which shows that $V(\zeta)$ is not bounded in general unless μ is atomic, in which case μ is decomposed into a CP-convex combination of pure CP-maps. In the following, we will discuss some classes of CP-maps which admit CP-pure decomposition, first in the sense of CP-convex combination, next in the sense of weak integral; this will give a partial solution to this problem.

We first identify the class of C*-algebras for which every CP-map $\psi \in CP(A, B(H))$ is decomposed into a CP-convex combination of pure CP-maps. We recall that a C*-algebra A is defined to be *scattered* if every state on A is represented by an atomic boundary measure, and a W*-algebra M is called *atomic* if the projection lattice of M is atomic, i.e., every projection of M contains a minimal projection in M. We recall that, due to H.E. Jensen ([48],[49]), the following conditions are equivalent.

(i) A is scattered.

(ii) A^{**} is atomic.

(iii) Every $\omega \in (A^*)^+$ is of the form $\omega = \sum_{i=1}^{\infty} \omega_i$ where ω_i is a pure positive linear functional on A.

(iv) Every non-degenerate representation of A is unitarily equivalent with a subresentation of a direct sum of irreducible representations.

(v) A is of type I and has scattered spectrum $\hat{A}$.

Remarks. 1. The condition (iii) can be expressed as $Q_{\mathbf{C}}(A) = \sigma - conv\, P_{\mathbf{C}}(A)_1$

where $P_{\mathbf{C}}(A)_1 := P_{\mathbf{C}}(A) \cap Q_{\mathbf{C}}(A)$,i.e., every quasi-state $\omega \in Q_{\mathbf{C}}(A)$ is of the form

$$\omega = \sum_{i=1}^{\infty} \lambda_i \omega_i \quad \text{with} \quad \sum_{i=1}^{\infty} \lambda_i = 1,\ \lambda_i > 0 \ \text{and} \ \ \omega_i \in P_{\mathbf{C}}(A)_1 .$$

2. Regarding condition (v), we note that a topological space T is called scattered if every closed subset of T has a point which is relatively isolated in T.

3. A separable C*-algebra A is scattered if and only if $\hat{A}$ is countable (cf.[48], [49]).

4. The C*-algebra $C(H)$ of all compact operators on a Hilbert space H is scattered, and the W*-algebra $B(H)$ is atomic.

In the setting of CP-convexity, a scattered C*-algebra is characterized as follows.

THEOREM 6.1. *Let A be a C*-algebra and H be a Hilbert space. Then A is a scattered C*-algebra if and only if the CP-state space $Q_H(A)$ is elementary, i.e.,*

$$Q_H(A) = CP - conv\, P_H(A)_1 \quad \text{where} \ P_H(A)_1 = P_H(A) \cap Q_H(A),$$

i.e., every CP-state $\psi \in Q_H(A)$ is represented of the form

$$\psi = \sum_{\alpha} S_\alpha^* \psi_\alpha S_\alpha \ \ \text{where} \ \psi_\alpha \in P_H(A)_1 \ \text{and} \ S_\alpha \in B(H) \ \text{such that} \ \sum_{\alpha} S_\alpha^* S_\alpha \leq I_H.$$

PROOF: Assume that A is a scattered C*-algebra, and let $\psi = V^*\pi V \in Q_H(A)$. By the above equivalent condition (iv), there exist a family $(\pi_\alpha)_{\alpha\in\Lambda} \subset Irr(A)$ such that $\pi = (\oplus_{\alpha\in\Lambda}\pi_\alpha)|_{H_\pi}$. Hence, letting p_α be the projection of $H_\pi = \oplus_{\alpha\in\Lambda} H_{\pi_\alpha}$ onto H_{π_α}, we have

$$\psi = V^*(\oplus_{\alpha\in\Lambda} p_\alpha)(\oplus_{\alpha\in\Lambda}\pi_\alpha)(\oplus_{\alpha\in\Lambda} p_\alpha)V = \sum_{\alpha\in\Lambda} V_\alpha^* \pi_\alpha V_\alpha,$$

where V_α is defined by $V_\alpha := p_\alpha V$ and $\sum_{\alpha\in\Lambda} V_\alpha^* V_\alpha = \sum_{\alpha\in\Lambda} V^* p_\alpha V = V^*V \leq I_H$. This shows that $Q_H(A)$ is elementary.

Conversely, suppose that $Q_H(A)$ is elementary, i.e., every $\psi \in Q_H(A)$ is written of the form

$$\psi = \sum_{\alpha\in\Lambda} S_\alpha^* \psi_\alpha S_\alpha \quad \text{where } \psi_\alpha = V_\alpha \pi_\alpha V_\alpha \in P_H(A)_1$$

$$\text{and } S_\alpha \in B(H) \text{ with } \sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha \leq I_H.$$

Recall that every $\omega \in Q_{\mathbf{C}}(A)$ is of the form $\omega = P\psi P$ with $\psi \in Q_H(A)$ where P is a one dimensional projection on H. Let $h \in im\, P$ with $\|h\| = 1$, then

$$\omega = (\psi h, h) = \sum_{\alpha\in\Lambda} (\pi_\alpha V_\alpha S_\alpha h, V_\alpha S_\alpha h) = \sum_{\alpha\in\Lambda} \omega_\alpha,$$

where we set $\omega_\alpha := (\pi_\alpha(\cdot) V_\alpha S_\alpha h, V_\alpha S_\alpha h)$ and note that the sum of the above is at most countable, since $0 \leq \omega(e) = \sum_{\alpha\in\Lambda} \omega_\alpha(e) \leq 1$. This shows that ω is atomic, hence A is scattered. ∎

Remarks. 1. The condition $Q_H(A) = CP - conv\, P_H(A)_1$ is equivalent to that every $\psi \in CP(A, B(H))$ is decomposed as $\psi = \sum_{\alpha\in\Lambda} \psi_\alpha$ where $\psi_\alpha \in P_H(A)$.

2. If H is separable, then the condition is expressed as $Q_H(A) = \sigma - CP - conv\, P_H(A)_1$ (i.e., $\mathrm{Card}(\Lambda) \leq \aleph_0$). Indeed, if H is separable, then $B(H)$ is σ-finite, so that with a faithful normal state ω_0 and $e \in A$ (or, if A is not unital, using an approximate identity (e_λ)), we have $0 < \omega_0(\psi(e)) = \omega_0(\sum_{\alpha\in\Lambda} S_\alpha^* \psi_\alpha(e) S_\alpha) \leq \omega_0(\sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha) \leq \omega_0(I_H) = 1$, so that $\sum_{\alpha\Lambda} S_\alpha^* S_\alpha$ is a countable sum on the faithful normal ω_0, hence Λ is countable.

3. K. Kraus [55] showed the above decomposition for the particular case $\psi \in CP(C(H), B(H)) = CP(B(H), B(H))_n := CP(B(H))_n$.

For the remainder of this chapter, we assume that A is a separable C*-algebra, and investigate the class of CP-maps $\psi = V^*\pi V \in CP(A, B(H))$ which can be disintegrated into pure CP-maps. (The separability condition on A will be needed for the irreducible decomposition of π.) As we shall see in the weak integral decomposition as follows, our assumption of a strong continuity on the operator V, in terms of a nuclear topology, will be essential in the subsequent arguments, so that we prepare some definitions and notations for nuclear spaces (cf. Appendix 6 for basic definitions).

Let $\Omega \subset H_\Omega \subset \Omega'$ be a rigged Hilbert space. Recall that the nuclear topology of Ω is defined by a chain of countably many Hilbert spaces Ω_n with norms $\|\cdot\|_n$ $(n = 1, 2, 3, \cdots)$, such that

$$\|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots \leq \|\cdot\|_n \leq \|\cdot\|_{n+1} \leq \cdots$$

and

$$\Omega \subset \cdots \subset \Omega_{n+1} \subset \Omega_n \subset \cdots \subset \Omega_2 \subset \Omega_1 \subset H_\Omega.$$

Since the norm $\|\cdot\|$ of H_Ω is continuous with respect to the nuclear topology on Ω by definition, there exists $n \in \mathbb{N}$ such that the embedding $T : \Omega \to H_\Omega$ is a nuclear operator with respect to some norm $\|\cdot\|_n$ (cf.[39]). We denote by $r(\Omega)$ the minimum of this integer n, i.e.,

$$r(\Omega) := \min\{n \in \mathbb{N};\ T : \Omega \hookrightarrow H_\Omega \text{ is a nuclear operator with respect to } \|\cdot\|_n\},$$

and call the *rank* of the rigged Hilbert space $\Omega \subset H_\Omega \subset \Omega'$. In the following, we will use the abbreviations $\Omega_r := \Omega_{r(\Omega)}$ and $\|\cdot\|_r := \|\cdot\|_{r(\Omega)}$, and we denote by c_r the operator norm of the nuclear embedding $T : \Omega_r \to H_\Omega$, i.e., $\|\cdot\| \leq c_r\|\cdot\|_r$.

DEFINITION 6.2. *Let A be a C^*-algebra, H be a Hilbert space and let $\psi = V^*\pi V \in CP(A, B(H))$ where π is a representation of A on H_π and $V \in B(H, H_\pi)$, and assume that there exists a countably Hilbert nuclear space Ω such that $\Omega \subset H_\pi \subset \Omega'$ is a rigged Hilbert space. Then, ψ is defined to be **n-continuous** if $V \in B(H, \Omega_n)$, i.e., V is a bounded linear operator from the Hilbert space H to the Hilbert space Ω_n with the $\|\cdot\|_n$- norm. We then define ψ to be **pre-nuclear** if ψ is $r(\Omega)$-continuous, and define ψ to be **nuclear** if ψ is n-continuous for all $n \in \mathbb{N}$, i.e., $V : H \to \Omega = \cap_{n=1}^{\infty}\Omega_n$ is continuous with respect to the nuclear topology.*

Remarks. 1. We note that, if ψ is n-continuous, then ψ is m-continuous for all $m < n$.

2. As an example of nuclear CP-maps, we cite the class of finite rank CP-maps,i.e., $\psi = V^*\pi V$ where $dim\, V(H) < \infty$. One can take a countably Hilbert nuclear space Ω such that $V(H) \subset \Omega$ and $\Omega \subset H_\Omega \subset \Omega'$ is a rigged Hilbert space.

3. If $\psi = V^*\pi V$ is pre-nuclear, then V is a nuclear (or trace class) operator from H to H_π, since V is decomposed as $V = T \circ V : H \xrightarrow{V} \Omega_r \xrightarrow{T} H_\pi$ where T is the nuclear embedding.

Now, we prove the following decomposition theorem for pre-nuclear CP-maps.

THEOREM 6.3. *Let A be a separable C^*-algebra and H be a separable Hilbert space with $dimH \geq \alpha_i(A) := \sup\{dimH_\pi; \pi$ is an irreducible representation of A on $H_\pi\}$. Let $\psi = V^*\pi V \in CP(A, B(H))$ where π is a representation of A on a separable Hilbert space H_π and assume that ψ is a pre-nuclear CP-map. Then, for any maximal abelian subalgebra $\mathcal{M}_\pi$ of $\pi(A)'$, there exists a standard measure space*

(Z,μ) which corresponds to $\mathcal{M}_\pi$, and measurable families $(\pi(\zeta))_{\zeta\in Z} \subset Irr(A:H)$ (the set of all irreducible representations of A on H) and $(V(\zeta))_{\zeta\in Z} \subset \mathcal{T}(H)$ (the set of all trace class operators on H) such that

$$\psi = \int_Z V(\zeta)^* \pi(\zeta) V(\zeta)\, d\mu(\zeta) \qquad \text{(BW-integral)}$$

i.e.,

$$(\psi(a), \rho) = \int_Z (V(\zeta)^* \pi(\zeta)(a) V(\zeta), \rho)\, d\mu(\zeta) \qquad \text{for all} \quad a \in A \quad \text{and} \quad \rho \in B(H)_* .$$

PROOF: From the standard disintegration theorem (e.g., [28;Th.8.5.2]), for a representation π of A on a separable Hilbert space H_π, and a maximal abelian subalgebra $\mathcal{M}_\pi$ of $\pi(A)'$, there exists a standard measure space (Z,μ) corresponding to $\mathcal{M}_\pi$, and a measurable family $(\pi(\zeta))_{\zeta\in Z}$ of irreducible representations of A on $(H_\pi(\zeta))_{\zeta\in Z}$ such that

$$\pi = \int_Z^{\oplus} \pi(\zeta)\, d\mu(\zeta) \qquad \text{and} \qquad H_\pi = \int_Z^{\oplus} H_\pi(\zeta)\, d\mu(\zeta).$$

In the above decomposition of π, H_π is assumed to be separable, hence there exists a countably Hilbert nuclear space Ω such that H_π is equipped as a rigged Hilbert space $\Omega \subset H_\pi \subset \Omega'$. Since H_π is represented as the direct integral of the Hilbert spaces $(H_\pi(\zeta))_{\zeta\in Z}$, we know from [39; §4.4, Th.1'] (cf. Appendix 6) that there exists a family of nuclear operators $\{T_\zeta : \Omega \to H_\pi(\zeta)\}_{\zeta\in Z}$ such that, for every $f \in \Omega$,

$$T_\zeta(f) = f(\zeta) \qquad \text{for} \quad \mu - a.e.\ \zeta \in Z.$$

From the arguments in [39], T_ζ is continuous with respect to $\|\cdot\|_r$-norm, so we define $P(\zeta) := \bar{T}_\zeta \in B(\Omega_r, H_\pi(\zeta))$ where the closure is taken by the norm $\|\cdot\|_r$. By the assumption that ψ is pre-nuclear, $V \in B(H, \Omega_r)$, so that we can define

$$V(\zeta) := P(\zeta)V : H \to H_\pi(\zeta),$$

and clearly $V(\zeta)$ is a nuclear operator from H to $H_\pi(\zeta)$, i.e., $V(\zeta) \in \mathcal{T}(H, H_\pi(\zeta)) \subset B(H, H_\pi(\zeta))$.

Noting that $V(\zeta)^*\pi(\zeta)(a)V(\zeta) \in B(H)$ for all $a \in A$, we set, for arbitrary $\rho \in B(H)_*$,

$$\Psi(a;\rho)(\zeta) := (V(\zeta)^*\pi(\zeta)(a)\,V(\zeta), \rho).$$

We will show that

$$\Psi(a;\rho)(\zeta) \in L^1(Z,\mu) \qquad \text{for all} \quad a \in A \quad \text{and} \quad \rho \in B(H)_*.$$

We then use this fact and the arguments given in [27;II.3 pp 52-53] to conclude that there exists $\Psi(a) \in B(H)$ such that

$$(\Psi(a), \rho) = \int_Z \Psi(a;\rho)(\zeta)\,d\mu(\zeta) \qquad \text{for all} \quad a \in A \quad \text{and} \quad \rho \in B(H)_*,$$

i.e., there exists $\Psi : A \to B(H)$ such that

$$\Psi = \int_Z V(\zeta)^*\pi(\zeta)\,V(\zeta)\,d\mu(\zeta) \qquad \text{(BW-integral)}.$$

After establishing this fact, we will show that $\Psi = \psi$, and that $\pi(\zeta)$ and $V(\zeta)$ can be taken in $Irr(A : H)$ and $\mathcal{T}(H)$ respectively.

We first consider the simplest case where ρ is of the form $\rho = \omega_{\xi,\eta}$ with $\xi, \eta \in H$, i.e.,

$$\rho(x) = \omega_{\xi,\eta}(x) = (x\xi, \eta) \qquad \text{for} \quad x \in B(H).$$

In this case,

$$\Psi(a;\rho)(\zeta) = (V(\zeta)^*\pi(\zeta)(a)\,V(\zeta)\,\xi,\eta) = (P(\zeta)^*\pi(\zeta)(a)P(\zeta)\,V\,\xi,V\eta).$$

We need to show that $\Psi(a;\rho)$ is a μ -measurable function of $\zeta \in Z$. Consider the functions, for $f,g \in \Omega$,

$$F(a;f,g)(\zeta) := (P(\zeta)^*\pi(\zeta)(a)\,P(\zeta)\,f,g) = (\pi(\zeta)(a)\,P(\zeta)\,f,P(\zeta)\,g)$$

$$= (\pi(\zeta)(a)\,f(\zeta),g(\zeta)) \qquad \mu - a.e.\ \zeta \in Z,$$

so that $F(a;f,g)(\zeta)$ is μ -measurable for all $f,g \in \Omega$. Since Ω is $\|\cdot\|_r$-norm dense in Ω_r, we can take sequences $(f_n),(g_m)$ in Ω which converges to $V\xi$ and $V\eta$ in $\|\cdot\|_r$-norm respectively. Then,

$$
\begin{aligned}
&|\Psi(a;\rho)(\zeta) - F(a;f_n,g_m)(\zeta)| \\
&= |(P(\zeta)^*\pi(\zeta)(a)\,P(\zeta)\,V\xi,V\eta) - (P(\zeta)^*\pi(\zeta)(a)\,P(\zeta)\,f_n,g_m)| \\
&\le |(P(\zeta)^*\pi(\zeta)(a)\,P(\zeta)\,(V\xi - f_n),V\eta)| + |(P(\zeta)^*\pi(\zeta)(a)\,P(\zeta)\,f_n,V\eta - g_m)| \\
&\le \|P(\zeta)^*\pi(\zeta)(a)\,P(\zeta)\,(V\xi - f_n)\|\|V\eta\| + \|P(\zeta)^*\pi(\zeta)(a)\,P(\zeta)\,f_n\|\|V\eta - g_m\| \\
&\le c_r^2\|P(\zeta)^*\pi(\zeta)(a)P(\zeta)\|_r(\|V\xi - f_n\|_r\|V\eta\|_r + \|f_n\|_r\|V\eta - g_m\|_r) \\
&\le c_r^2\|P(\zeta)\|_r^2\|a\|^2(\|V\xi - f_n\|_r\|V\eta\|_r + \|f_n\|_r\|V\eta - g_m\|_r),
\end{aligned}
$$

where the subscript r to $\|P(\zeta)\|_r$ denotes the operator norm with respect to the $\|\cdot\|_r$-norm in Ω_r and the norm in $H_\pi(\zeta)$, and recall that $\|h\| \le c_r\|h\|_r$ for $h \in \Omega_r$. Now, (f_n) converges to $V\xi$ in Ω_r, hence $(\|f_n\|_r)$ is uniformly bounded, so $F(a{:}\,f_n,g_m)(\zeta)$ converges pointwise to $\Psi(a;\rho)(\zeta)$ $(\zeta \in Z)$ as $n \to \infty$, $m \to \infty$. Hence $\Psi(a;\rho)(\zeta)$ is indeed μ -measurable.

For $\rho = \omega_{\xi,\eta}$ with $\xi, \eta \in H$, we now prove that $\Psi(a;\rho)(\zeta) \in L^1(Z,\mu)$ and that one has in fact

$$\|\Psi(a;\rho)\|_{L^1} \leq c_r^2 \|a\| \|V\|_r^2 \|\xi\| \|\eta\|,$$

where $\|V\|_r$ denotes the operator norm wirh respect to the norm $\|\cdot\|$ in H and the $\|\cdot\|_r$-norm in Ω_r. To do so, we observe that, for $f, g \in \Omega$,

$$\begin{aligned}
\|F(a;f,g)\|_{L^1} &= \int_Z |F(a;f,g)(\zeta)|\, d\mu(\zeta) = \int_Z |((\pi(a)f)(\zeta), g(\zeta))|\, d\mu(\zeta) \\
&\leq \int_Z \|(\pi(a)f)(\zeta)\| \|g(\zeta)\|\, d\mu(\zeta) \\
&\leq (\int_Z \|(\pi(a)f)(\zeta)\|^2\, d\mu(\zeta))^{\frac{1}{2}} (\int_Z \|g(\zeta)\|^2\, d\mu(\zeta))^{\frac{1}{2}} \\
&= \|\pi(a)f\| \|g\| \leq \|a\| \|f\| \|g\| \\
&= c_r^2 \|a\| \|f\|_r \|g\|_r .
\end{aligned}$$

With now (f_n) [resp. (g_m)] converging to $V\xi$ [resp. $V\eta$] in Ω_r, by Fatou's Lemma, we have

$$\begin{aligned}
\|\Psi(a;\rho)\|_{L^1} &\leq \varliminf_{n,m\to\infty} \|F(a;f_n,g_m)\|_{L^1} \leq \varliminf_{n,m\to\infty} c_r^2 \|a\| \|f_n\|_r \|g_m\|_r \\
&= c_r^2 \|a\| \|V\xi\|_r \|V\eta\|_r \leq c_r^2 \|a\| \|V\|_r^2 \|\xi\| \|\eta\|.
\end{aligned}$$

We are now ready to consider the general case when ρ is arbitrary in $B(H)_*$, i.e.,

$$\rho(x) = \sum_{n=1}^{\infty} \omega_{\xi_n,\eta_n}(x) = \sum_{n=1}^{\infty} (x\xi_n, \eta_n) \qquad \text{for } x \in B(H)$$

$$\text{where} \quad \sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty.$$

It suffices to observe that

$$\begin{aligned}\sum_{n=1}^{\infty}\|\Psi(a;\omega_{\xi_n,\eta_n})\|_{L^1} &\leq c_r^2\,\|a\|\|V\|_r^2\sum_{n=1}^{\infty}\|\xi_n\|\|\eta_n\| \\ &\leq c_r^2\,\|a\|\|V\|_r^2(\sum_{n=1}^{\infty}\|\xi_n\|^2)^{\frac{1}{2}}(\sum_{n=1}^{\infty}\|\eta_n\|^2)^{\frac{1}{2}} < \infty,\end{aligned}$$

from which we conclude that

$$\Psi(a;\rho)(\zeta) = \sum_{n=1}^{\infty}\Psi(a;\omega_{\xi_n,\eta_n})(\zeta) \in L^1(Z,\mu) \qquad \text{for all} \quad a \in A \quad \text{and} \quad \rho \in B(H)_*.$$

As we mentioned before, this ensures the existence of $\Psi : A \to B(H)$ such that

$$\Psi = \int_Z V(\zeta)^*\pi(\zeta)\,V(\zeta)\,d\mu(\zeta) \qquad \text{(BW-integral).}$$

We shall show that Ψ defined above is actually equal to our original ψ. Note first that, for all $a \in A$ and $\xi, \eta \in H$,

$$(\psi(a) - \Psi(a), \omega_{\xi,\eta}) = (V^*\pi(a)\,V\xi, \eta) - \int_Z (P(\zeta)^*\pi(\zeta)(a)\,P(\zeta)\,V\xi, V\eta)\,d\mu(\zeta).$$

Let us consider

$$G(a;h,k) := (\pi(a)h,k) - \int_Z (P(\zeta)^*\pi(\zeta)(a)\,P(\zeta)\,h,k)\,d\mu(\zeta) \qquad \text{for } h,k \in \Omega_r.$$

If $h = f \in \Omega$ and $k = g \in \Omega$, then

$$G(a;h,k) = (\pi(a)f,g) - \int_Z (\pi(\zeta)(a)f(\zeta), g(\zeta))\,d\mu(\zeta) = 0.$$

Since $G(a;h,k)$ is a $\|\cdot\|$-bounded, hence $\|\cdot\|_r$-bounded sesquilinear functional on $\Omega \times \Omega$ which is dense in $\Omega_r \times \Omega_r$, we conclude $G(a;h,k) = 0$ for all $a \in A$ and $h, k \in \Omega_r$. Let $h = V\xi$ and $k = V\eta$, then $(\psi(a) - \Psi(a), \omega_{\xi,\eta}) = G(a; V\xi, V\eta) = 0$.

Since $\xi, \eta \in H$ are arbitrary, we conclude that $\psi(a) = \Psi(a)$ for all $a \in A$, i.e., $\psi = \Psi$. Hence we have proved the decomposition of ψ, i.e.,

$$\psi = \int_Z V(\zeta)^* \pi(\zeta) V(\zeta)\, d\mu(\zeta) \qquad \text{(BW-integral)}.$$

To complete the proof of Theorem 6.3, we have to show that we can take $\pi(\zeta) \in Irr(A : H)$ and $V(\zeta) \in \mathcal{T}(H)$ without loss of generality. By assumption, $dim\, H \geq dim\, H_\pi(\zeta)$ for all $\zeta \in Z$, so that there exists a partial isometry $W(\zeta) : H \to H_\pi(\zeta)$ from H onto $H_\pi(\zeta)$, i.e., $W(\zeta)W(\zeta)^* = id_{H_\pi(\zeta)}$. Then,

$$\begin{aligned}\psi &= \int_Z V(\zeta)^* W(\zeta) W(\zeta)^* \pi(\zeta) W(\zeta) W(\zeta)^* V(\zeta)\, d\mu(\zeta) \\ &= \int_Z (W(\zeta)^* V(\zeta))^* (W(\zeta)^* \pi(\zeta) W(\zeta))(W(\zeta)^* V(\zeta))\, d\mu(\zeta),\end{aligned}$$

where $W(\zeta)^* V(\zeta) \in \mathcal{T}(H)$ since $V(\zeta) \in \mathcal{T}(H, H_\pi(\zeta))$, and $W(\zeta)^* \pi(\zeta) W(\zeta) \in Irr(A : H)$ since $W(\zeta)$ is a partial isometry from H onto $H_\pi(\zeta)$ and $\pi(\zeta)$ is an irreducible representation of A on $H_\pi(\zeta)$. This completes the proof of Theorem 6.3. ∎

Remark. The condition on the dimension of the Hilbert space H is not necessary if H is infinite dimensional. However it diminishes the dimension of H if A is finite dimensional.

Based on the CP-decomposition theorems which we discussed in Theorems 6.1 and 6.4, we shall generalize Choquet's theorem for CP-state spaces.

THEOREM 6.4. *Let A be a separable scattered C^*-algebra and H be a Hilbert space with $dim\, H \geq \alpha_i(A)$. Then, for every $\psi \in Q_H(A)$, there exists a $Q_H(B(H))_n$-*

valued CP-measure λ_ψ *which is supported by* $Irr(A:H)$ *such that*

$$\psi = b(\lambda_\psi) := \int_{\varphi \in Q_H(A)} \varphi \, d\lambda_\psi,$$

where the integral is the CP-integral (in the strong or weak sense) and $b(\psi)$ *is defined to be the barycenter of the CP-measure* λ_ψ *.*

PROOF: It follows from Theorem 6.1 and the condition $dim H \geq \alpha_i(A)$ that every $\psi \in Q_H(A)$ is decomposed as

$$\psi = \sum_\alpha V_\alpha^* \pi_\alpha V_\alpha \quad \text{where } \pi_\alpha \in Irr(A:H) \text{ and } V_\alpha \in B(H)$$

$$\text{such that } \sum_\alpha V_\alpha^* V_\alpha \leq I_H .$$

Let $X = Q_H(A)$ and let $\mathcal{B}_s$ denote the Borel sets induced from the norm topology of $Q_H(A)$. We define

$$\lambda_\psi : E \in \mathcal{B}_s \mapsto \lambda_\psi(E) = \sum_{\pi_\alpha \in E} V_\alpha(\cdot) V_\alpha \in Q_H(B(H))_n,$$

which is clearly a BW-countably additive $Q_H(B(H))_n$-valued CP-measure. We can regard $a \in BM_s(X, B(H))$ for every $a \in A$. Hence, from Definition 5.5, we can define the following CP-integral in the strong sense

$$\int_{\varphi \in Q_H(A)} a(\varphi) \, d\lambda_\psi = \int_{\varphi \in Q_H(A)} \varphi(a) \, d\lambda_\psi \; (:= b(\lambda_\psi)).$$

We shall show that $\psi = b(\lambda_\psi)$. We first note that, in the CP-decomposition $\psi = \sum_\alpha V_\alpha^* \pi_\alpha V_\alpha$, if $\pi_\beta \simeq \pi_\alpha$, then we can set $\pi_\beta = U_{\alpha\beta}^* \pi_\alpha U_{\alpha\beta}$ for some partial isometry $U_{\alpha\beta}$ in H, so that we can consider that $\cup_\alpha a(\pi_\alpha) \subset \cup_\alpha a(Irr(A:H)\hat{})$ where $\hat{}$ denotes the equivalence class. Let us define

$$C_H^*(\hat{A}) := \text{the C*-algebra generated by } \cup \{\pi(A); \pi \in Irr(A:H)\hat{}\}.$$

Since A is separable and scattered, $\pi(A)$ is separable and $\mathrm{Card}(Irr(A : H)^\wedge) = \mathrm{Card}(\hat{A}) = \aleph_0$, so that $C^*_H(\hat{A})$ is a separable C*-algebra. We can consider that a is $C^*_H(\hat{A})$-valued on the measurable set $Irr(A : H)^\wedge$ (countable) and we can assume $a > 0$ without loss of generality. Then, by Proposition 5.7, there exists a monotone increasing sequence $(f_k)_{k=1}^\infty \subset BS(X, C^*_H(\hat{A}))$ such that $f_k \nearrow a$ uniformly on the support of λ_ψ and $\int a\, d\lambda_\psi = \lim_{k\to\infty} \int f_k\, d\lambda_\psi$.

Let us express

$$f_k = \sum_{i=1}^{\infty} \chi_{E_i^{(k)}} b_i^{(k)} \quad \text{with } (E_i^{(k)}) \subset \mathcal{B}_s \text{ disjoint and } (b_i^{(k)})_{i=1}^\infty \subset C^*_H(\hat{A})^+,$$

and observe that

$$\begin{aligned}
\|\psi(a) - \int f_k\, d\lambda_\psi\| &= \|\sum_\alpha V_\alpha^* \pi_\alpha(a) V_\alpha - \sum_{i=1}^{\infty} \lambda_\psi(E_i^{(k)}) b_i^{(k)}\| \\
&= \|\sum_\alpha V_\alpha^* \pi_\alpha(a) V_\alpha - \sum_{i=1}^{\infty} \sum_{\pi_\alpha \in E_i^{(k)}} V_\alpha^* b_i^{(k)} V_\alpha\| \\
&= \|\sum_\alpha V_\alpha^* (\pi_\alpha(a) - f_k(\pi_\alpha)) V_\alpha\| \\
&\le \sup_\alpha \|\pi_\alpha(a) - f_k(\pi_\alpha)\| \|\sum_\alpha V_\alpha I_H V_\alpha\| \\
&\le \sup\{\|(a - f_k)(\varphi)\| \,;\, \varphi \in \text{ support of } \lambda_\psi\}.
\end{aligned}$$

Since $f_k \nearrow a$ uniformly on the support of λ_ψ, we conclude that

$$\psi(a) = \lim_{k\to\infty} \int f_k\, d\lambda_\psi = \int_{\varphi \in Q_H(A)} a(\varphi) d\,\lambda_\phi = \int_{\varphi \in Q_H(A)} \varphi(a)\, d\lambda_\psi.$$

in the strong sense.

For the weak sense, we replace $\mathcal{B}_s$ by $\mathcal{B}_{BW}$ which is the Borel sets induced by the BW-topology on $X = Q_H(A)$, and define the CP-measure λ_ψ similarly as in the strong case, which is clearly BW-countably additive $Q_H(B(H))_n$-valued CP-measure. We can consider that $a \in A$ is $C^*_H(\hat{A})^{**}$-valued function on the support of λ_ψ where $C^*_H(\hat{A})^{**}$ is w*-separable, and moreover λ_ψ is BW-absolutely continuous with respect to an atomic measure as we discussed above since A is separable and scattered. Thus, by Proposition 5.14, we can define the CP-integral of a with respect to λ_ψ in the weak sense, and it follows from the remark to the proposition that it coinsides with the CP-decomposition of ψ. This completes the proof. ∎

For general separable C*-algebras, we have to restrict our objective to pre-nuclear CP-maps.

THEOREM 6.5. *Let A be a separable C*-algebra and H be a separable Hilbert space with $\dim H \geq \alpha_i(A)$. Then, for every pre-nuclear CP-map $\psi = V^*\pi V \in Q_H(A)$, there exists a $Q_H(B(H))_n$-valued CP-measure λ_ψ which is supported by $Irr(A : H)$ such that*

$$\psi = b(\lambda_\psi) := \int_{\varphi \in Q_H(A)} \varphi \, d\lambda_\psi,$$

where the integral is the CP-integral in the weak sense.

PROOF: By Theorem 6.3, there exists a standard measure space (Z, μ) and measurable families $(\pi(\zeta))_{\zeta \in Z} \subset Irr(A : H)$ and $(V(\zeta))_{\zeta \in Z} \subset \mathcal{T}(H)$ such that

$$\psi = \int_Z V(\zeta)^* \pi(\zeta) V(\zeta) \, d\mu(\zeta) \qquad \text{(BW-integral)}.$$

Let $(X, \mathcal{B}_{BW})$ be the measurable space as we defined in the proof of Theorem 6.4. In view of the embedding

$$\int_Z^{\oplus} H_\pi(\zeta) \, d\mu(\zeta) \subset L^2(Z, \mu, B(H)),$$

we define a CP-measure $\lambda_\psi : E \in \mathcal{B}_{BW} \mapsto \lambda_\psi(E) \in Q_H(B(H))_n$ by

$$\lambda_\psi(E)(b) = \int_{\pi(\zeta)\in E}^{\oplus} V(\zeta)^* \, b \, V(\zeta) \, d\mu(\zeta) \quad \text{for } b \in B(H).$$

Since H is separable, $B(H)$ is w*-separable, and clearly λ_ψ is BW-absolutely continuous with respect to μ; hence by Proposition 5.15 we can define the CP-integral

$$\int_{\varphi\in Q_H(A)} a(\varphi)\, d\lambda_\psi = \int_{\varphi\in Q_H(A)} \varphi(a)\, d\lambda_\psi,$$

which, by the remark to the proposition, is shown to be equal to the BW-integral

$$\psi(a) = \int_Z V(\zeta)^* \pi(\zeta)(a) V(\zeta)\, d\mu(\zeta).$$

This completes the proof. ∎

CHAPTER 7

CP-ORIENTABILITY

We revisit the study of the CP-affine geometry of the cone of completely positive maps $CP(A, B(H)_{sa})$ for a JC-algebra A. In this chapter, we focus our attention on the condition which allows A to be the self-adjoint part of a C*-algebra; this is naturally motivated from purely mathematical interests as well as its physical meaning.

The conditions which characterize the category of C*- [resp. W*-] algebras among JB- [resp. JBW-] algebras are called orientability conditions. Alfsen, Hanche-Olsen and Shultz [4] first formulated this condition for C*-algebras in terms of facial structure of the state space, i.e., 3-ball property and its global orientability, and in this direction, Iochum and Shultz [47] characterized the normal state space of W*-algebras. On the other hand, from a different approach in connection with Tomita-Takesaki theory, Connes [24] (cf. also Bellisard and Iochum [13] and Iochum [46]) gave another orientability condition in terms of the facial structure of the self-dual cone of Hilbert space. The present chapter is intended to formulate the orientability condition in the CP-geometrical context, and to examine its relation with the orientability condition of the state space in [4]. We shall discuss Connes' orientability condition and its relation to Tomita-Takesaki theory in our CP-geometrical setting in Chapter 8.

In this chapter, we are mainly concerned with *universally reversible* JC-algebras (see Appendix 3 for definition). For our subsequent arguments, it shall be enough to note the following characterization of universally reversible JC-algebras. Recall that, for a JC-algebra A, there exists a unique *-anti-automorphism Φ (of order 2, i.e., $\Phi^2 = id$) of the enveloping C*-algebra $C^*(A)$ which leaves A pointwise invariant. Then, A is universally reversible if and only if $A = C^*(A)^{\Phi}_{sa} := \{a \in C^*(A);\ \Phi(a) = a\}$ (cf. [44; Lemma 4.2]).

LEMMA 7.1.A. *Let A_1 and A_2 be universally reversible JC-algebras. If $A_1 \subset A_2$ and $C^*(A_1) = C^*(A_2)$, then $A_1 = A_2$.*

LEMMA 7.1.B. *Let M_1 and M_2 be universally reversible JW-algebras. If $M_1 \subset M_2$ and $W^*(M_1) = W^*(M_2)$, then $M_1 = M_2$.*

PROOF: Part A : Let Φ_1 and Φ_2 be the canonical *-anti-automorphisms of $C^*(A_1) = C^*(A_2)$ leaving A_1 and A_2 pointwise invariant respectively. Since $A_1 \subset A_2$, Φ_2 leaves A_1 pointwise invariant. From the uniqueness of the canonical *-anti-automorphism for JC-algebras ([4;Cor. 5.2]), we conclude $\Phi_1 = \Phi_2$. Since A_1 and A_2 are universally reversible, $A_1 = C^*(A_1)^{\Phi_1}_{sa} = C^*(A_2)^{\Phi_2}_{sa} = A_2$.

Part B : The uniqueness of the canonical *-anti-automorphism and the equality $M = W^*(M)^{\Phi}_{sa}$ for universally reversible JW-algebra M are proved similarly for JC-algebra case due to [45; Th. 7.1.9] and [44; Lemma 4.2]. So, we can apply the same argument as in Part A. ∎

In the following, we will denote A, M for JC-, JW-algebras, and $\mathcal{A}$, $\mathcal{M}$ for C*-, W*-algebras.

DEFINITION 7.2. *Let $\mathcal{A}$ be a C^*-algebra and let H be a Hilbert space, and let $\pi : \mathcal{A} \to B(H_\pi)$ be a *-representation. Then, we define the **transposed representation** π^t of π by the *-anti-homomorphism $\pi^t : a \in \mathcal{A} \to \pi(a)^\circ = \pi(a^\circ) \in B(H_\pi)^\circ$ (cf. Appendix for the definition of the opposite algebras $\mathcal{A}^\circ$ and $B(H_\pi)^\circ$).*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, leftrightarrow] \arrow[d, "\pi"'] \arrow[dr, "\pi^t"] & \mathcal{A}^\circ \arrow[d, "\pi"] \\
B(H_\pi) \arrow[r, leftrightarrow] & B(H_\pi)^\circ
\end{tikzcd}$$

*If $\psi = V^*\pi V \in Q_H(\mathcal{A})$, we use the notation $\psi^t = V^*\pi^t V$ and denote the **transposed CP-state space** of $\mathcal{A}$ by $Q_H^t(\mathcal{A}) := \{\psi^t ; \psi \in Q_H(\mathcal{A})\}$.*

*A CP-affine function $\gamma : Q_H(\mathcal{A}) \to B(H)$ is defined to be **transposed** if the product is defined by*

$$(\gamma_1 \cdot \gamma_2)(\pi) = \gamma_2(\pi) \cdot \gamma_1(\pi) \qquad \text{for all } \pi \in Rep_c(\mathcal{A} : H).$$

We denote the set of all transposed BW-w continuous [resp. bounded] CP-affine functions from $Q_H(\mathcal{A})$ to $B(H)$ by $A^tC(Q_H(\mathcal{A}), B(H))$ [resp. $A^tB(Q_H(\mathcal{A}), B(H))$].

We note that $\psi^t \notin Q_H(\mathcal{A})$ in the C*-algebra sense, however $\psi^t \in Q_H(\mathcal{A}_{sa})$ in the JC-algebra sense. We will show in the following that $\psi^t \in Q_H(\mathcal{A}^\circ)$.

PROPOSITION 7.3.A. *Let $\mathcal{A}$ be a C^*-algebra and H be a Hilbert space with $dim H \geq \alpha_c(\mathcal{A})$. Then,*

$$\mathcal{A}^\circ \cong AC(Q_H^t(\mathcal{A}), B(H)) \cong A^tC(Q_H(\mathcal{A}), B(H)) \qquad \text{(*-isomorphism)}$$

PROPOSITION 7.3.B. *Let $\mathcal{M}$ be a W^*-algebra and H be a Hilbert space with $dimH \geq \alpha_c(\mathcal{M})_n$. Then,*

$$\mathcal{M}^\circ \cong AB(Q^t_H(\mathcal{M})_n, B(H)) \cong A^tB(Q_H(\mathcal{M})_n, B(H)) \qquad (\text{*-isomorphism})$$

PROOF: It is enough to prove Part A. From the CP-duality for $\mathcal{A}$ proved in Theorem 2.2.A, we have only to check the algebraic isomorphism. Observe that $\pi^t((ab)^\circ) = \pi^t(b^\circ a^\circ) = \pi^t(a^\circ)\pi^t(b^\circ)$ for all $a, b \in \mathcal{A}$, i.e., $\pi^t((\cdot)^\circ) : \mathcal{A} \mapsto B(H)$ is a $*$-representation of $\mathcal{A}$. Hence the function defined by

$$\gamma : \psi^t = V^*\pi^t V \in Q^t_H(\mathcal{A}) \longmapsto \psi^t(a^\circ) = V^*\pi^t(a^\circ)V \in B(H)$$

is a CP-affine function satisfying the product rule defined in Proposition 2.2.A, i.e., if $\psi^t = V^*\pi^t V \in Q^t_H(\mathcal{A}) = \sum_\alpha V^*_\alpha \pi^t_\alpha V_\alpha$ where $\pi^t_\alpha \in Rep^t_c(\mathcal{A} : H)$ with $\pi^t \simeq \oplus_\alpha \pi^t_\alpha$ and $V_\alpha \in B(H)$ such that $\sum_\alpha V^*_\alpha V_\alpha \leq I_H$, then for $\gamma_1, \gamma_2 \in AC(Q^t_H(\mathcal{A}), B(H))$,

$$(\gamma_1 \cdot \gamma_2)(\psi^t) := \sum_\alpha V^*_\alpha \gamma_1(\pi^t_\alpha) \cdot \gamma_2(\pi^t_\alpha) V_\alpha.$$

Thus, we proved $\mathcal{A}^\circ \simeq AC(Q^t_H(\mathcal{A}), B(H))$.

The isomorphism $\mathcal{A}^\circ \simeq A^tC(Q_H(\mathcal{A}), B(H))$ is clear since the definition of $A^tC\ (Q_(\mathcal{A}), B(H))$ is same as $AC(Q_H(\mathcal{A}), B(H))$ except the order of the product, which should be isomorphic to $\mathcal{A}^\circ$. ∎

Remark. From the above propositions, we have

$$Q_H(\mathcal{A}^\circ) = Q^t_H(\mathcal{A}) \qquad \text{and} \qquad Q_H(\mathcal{M}^\circ)_n = Q^t_H(\mathcal{M})_n.$$

It also shows that the transposed CP-affine functions $\gamma \in A^tC(Q_H(\mathcal{A}), B(H))$ are precisely the maps

$$\gamma : \psi = V^*\pi V \in Q_H(\mathcal{A}) \mapsto \psi(a^\circ) = \psi^t(a) = V^*\pi^t(a)V \in B(H).$$

Now we are ready to formulate our orientability condition in the CP-state space of the enveloping C*- [resp. W*-] algebras for universally reversible JC- [resp. JW-] algebras. As we defined in Chapter 4, the abelian [resp. non-abelian] part of the CP-state space $Q_H(\mathcal{A})$ of a C*-algebra $\mathcal{A}$ is defined by $Q_H(\mathcal{A})_{ab} := Q_H(\mathcal{A}^{**})_n^{ab}$ and $Q_H(\mathcal{A})_{nab} := Q_H(\mathcal{A}^{**})_n^{nab}$ (cf. Cor. 4.6.(iii)).

THEOREM 7.4.A. *Let A be a universally reversible JC-algebra and let H be a Hilbert space with $dim H \geq \alpha_c(A)$. Then, A is the self-adjoint part of some C*-algebra $\mathcal{A}$ if and only if*

$$Q_H(C^*(A)) = Q_H(\mathcal{A})_{nab} \oplus_{CP} Q_H^t(\mathcal{A})_{nab} \oplus_{CP} Q_H(\mathcal{A})_{ab}.$$

THEOREM 7.4.B. *Let M be a universally reversible JW-algebra and H be a Hilbert space with $dim H \geq \alpha_c(M)_n$. Then, M is a self-adjoint part of some W*-algebra $\mathcal{M}$ if and only if*

$$Q_H(W^*(M))_n = Q_H(\mathcal{M})_n^{nab} \oplus_{CP} Q_H^t(\mathcal{M})_n^{nab} \oplus_{CP} Q_H(\mathcal{M})_n^{ab}$$

PROOF: We first prove Part B. Assume $M = \mathcal{M}_{sa}$ for some W*-algebra $\mathcal{M}$. Then by [45; 7.4.15],

$$W^*(M) = \{a \oplus b^\circ \in \mathcal{M} \oplus \mathcal{M}^\circ ; a - b \in \mathcal{M}_{nab}\},$$

or equivalently,

$$W^*(M) = \mathcal{M}_{nab} \oplus \mathcal{M}^\circ_{nab} \oplus \mathcal{M}_{ab}.$$

Let $\psi = V^*\pi V \in Q_H(W^*(M)_n)$ where π is a normal *-representation of $W^*(M)$ on a Hilbert space H_π, and $V \in B(H, H_\pi)$. We can set $\pi = \pi_1 \oplus \pi_2 \oplus \pi_3$ where π_i

$(i = 1, 2, 3)$ is a normal *-representation of $\mathcal{M}_{nab}$, $\mathcal{M}^{\circ}_{nab}$ and $\mathcal{M}_{ab}$ on a Hilbert space H_{π_i} respectively with $H_{\pi} = \oplus_{i=1}^{3} H_{\pi_i}$, and set $V = V_1 \oplus V_2 \oplus V_3$ with $V_i \in B(H, H_{\pi_i})$ $(i = 1, 2, 3)$ satisfying $V^*V = \sum_{i=1}^{3} V_i^* V_i \leq I_H$. Then for any $a \oplus b^{\circ} \oplus c \in W^*(M)$ with $a, b \in \mathcal{M}_{nab}, c \in \mathcal{M}_{ab}$, we have

$$\begin{aligned}\psi &= (\oplus_{i=1}^{3} V_i)^* (\oplus \pi_i)(a \oplus b^{\circ} \oplus c)(\oplus_{i=1}^{3} V_i) \\ &= V_1^* \pi_1(a) V_1 + V_2^* \pi_2^t(b) V_2 + V_3^* \pi_3(c) V_3.\end{aligned}$$

Since π_i and V_i are uniquely determined by ψ, we have the desired CP-split decomposition.

Conversely, assume the prescribed CP-decomposition, i.e.,

$$\begin{aligned}Q_H(W^*(M))_n &= Q_H(\mathcal{M})_n^{nab} \oplus_{CP} Q_H^t(\mathcal{M})_n^{nab} \oplus_{CP} Q_H(\mathcal{M})_n^{ab} \\ &= Q_H(\mathcal{M}_{nab})_n \oplus_{CP} Q_H^t(\mathcal{M}_{nab})_n \oplus_{CP} Q_H(\mathcal{M}_{ab})_n\end{aligned}$$

Then, by the CP-duality for $W^*(M)$, and for $\mathcal{M}^{\circ}$ (Proposition 7.3.B), we have

$$W^*(M) = \mathcal{M}_{nab} \oplus \mathcal{M}^{\circ}_{nab} \oplus \mathcal{M}_{ab} = W^*(\mathcal{M}_{sa}),$$

where we used again [45; 7.4.15] in the last equality. Clearly, $M \subset \mathcal{M}_{sa}$ from the expression of $Q_H(W^*(M))_n$. Hence, by Lemma 7.1.B, we conclude $M = \mathcal{M}_{sa}$.

Part A follows from Part B, since A is the self-adjoint part of some C*-algebra $\mathcal{A}$ if and only if its enveloping JW-algebra A^{**} is the self-adjoint part of the W*-algebra $\mathcal{A}^{**}$, and since $C^*(A)^{**} = W^*(A^{**})$ (cf. [45;7.1.11]). ∎

Remark. From Propositions 7.3 and 7.4, we immediately deduce that, for universally reversible JC-algebra $\mathcal{A}$ [resp. JW-algebra $\mathcal{M}$], there exist *-isomorphisms $C^*(\mathcal{A}) \cong C^*(\mathcal{A})^{\circ}$ and $W^*(M) \cong W^*(M)^{\circ}$.

Theorem 7.4.A [resp. B] gives an orientability condition in $Q_H(C^*(A))$ [resp. $Q_H(W^*(M))_n$]. To get a simpler formulation of the condition directly in the CP-state space $Q_H(A)$ [resp. $Q_H(M)_n$], we make a closer study of the connection between $Q_H(C^*(A))$ and $Q_H(A)$ [resp. $Q_H(W^*(M))_n$ and $Q_H(M)_n$]. The following is a direct consequence of Theorem 7.4.A and B.

COROLLARY 7.5.A. *Let A be a JC-algebra and H be a Hilbert space with $dim H \geq \alpha_c(A)$. If A is the self-adjoint part of some C^*-algebra $\mathcal{A}$, then for any $\psi \in Q_H(A)$, there exist *-representations π_1, π_2 of $\mathcal{A}$ and $V_i \in B(H, H_{\pi_i})$ $(i = 1, 2)$ with $\sum_{i=1}^{2} V_i^* V_i \leq I_H$, such that ψ is written as*

$$\psi = V_1^* \pi_1 V_1 + V_2^* \pi_2^t V_2.$$

COROLLARY 7.5.B. *Let M be a JW-algebra and H be a Hilbert space with $dim H \geq \alpha_c(M)_n$. If M is the self-adjoint part of some W^*-algebra $\mathcal{M}$, then for any $\psi \in Q_H(M)_n$, there exist normal *-representations π_1, π_2 of $\mathcal{M}$ and $V_i \in B(H, H_{\pi_i})$ $(i = 1, 2)$ with $\sum_{i=1}^{2} V_i^* V_i \leq I_H$, such that ψ is written as*

$$\psi = V_1^* \pi_1 V_1 + V_2^* \pi_2^t V_2.$$

PROOF: It is enough to prove Part A, since the Part B is obtained from the same arguments taking the normal part.

From Theorem 7.4.A, there exist non-abelian representations κ_1,κ_2 of $\mathcal{A}$ and an abelian representation κ_0, and $W_i \in B(H, H_{\kappa_i})$ $(i = 0, 1, 2)$ with $\sum_{i=0}^{2} W_i^* W_i \leq I_H$, such that ψ is written as

$$\psi = W_1^* \kappa_1 W_1 + W_2^* \kappa_2^t W_2 + W_0^* \kappa_0 W_0.$$

Set

$$\pi_1 = \kappa_1 \oplus \kappa_0 \qquad \text{and} \qquad V_1 = W_1 \oplus \frac{1}{\sqrt{2}} W_0$$

$$\pi_2 = \kappa_2 \oplus \kappa_0 \qquad \text{and} \qquad V_2 = W_2 \oplus \frac{1}{\sqrt{2}} W_0.$$

Then,

$$\psi = V_1^* \pi_1 V_1 + V_2^* \pi_2^t V_2 \qquad \text{with} \qquad \sum_{i=1}^{2} V_i^* V_i = \sum_{i=0}^{2} W_i^* W_i \leq I_H.$$

∎

Unfortunately, the above expression is not unique in general. We give an counter example to illustrate this situation. To simplify our argument, we assume that a JC-algebra A is the self-adjoint part of a C*-algebra $\mathcal{A}$ with no one dimensional representation, hence $C^*(A) = \mathcal{A} \oplus \mathcal{A}^\circ$ (cf. Theorem 7.4.A).

Let π be a cyclic representation of $\mathcal{A}$ with cyclic vector ξ ($\|\xi\| = 1$), and define $V \in B(H, H_\pi)$ by $Vk = (k,h)\xi$ for a fixed $h \in H$ with $\|h\| = 1$ and for all $k \in H$. We then define $\psi_i = V_i^* \kappa_i V_i \in Q_H(C^*(\mathcal{A}))$ $(i = 1,2)$ where κ_i is the *-representation on $H_\pi \oplus H_\pi$ defined by $\kappa_1 = \pi \oplus 0$ and $\kappa_2 = 0 \oplus \pi$, and $V_i \in B(H, H_\pi \oplus H_\pi)$ is defined by $V_1 = V \oplus 0$ and $V_2 = 0 \oplus V$ respectively. By the same argument in Proposition 1.5, for any $a \oplus b^\circ \in C^*(A)$, we have

$$\psi_1(a \oplus b^\circ) = V_1^* \kappa_1(a \oplus b^\circ) V_1 = V^* \pi(a) V = (\pi(a)\xi, \xi) P_h,$$

$$\psi_2(a \oplus b^\circ) = V_2^* \kappa_2(a \oplus b^\circ) V_2 = V^* \pi(b^\circ) V = (\pi^t(b)\xi, \xi) P_h.$$

We assume that the transposed representation π^t is given by $\pi^t = J_\pi \pi^* J_\pi$ where J_π is an antiunitary operator on H_π, and we assume here, as a particular case, that

π and J_π are chosen so that J_π leaves ξ invariant, i.e., $J_\pi\xi = \xi$. (If π is irreducible, this requirement is always possible.) Then,

$$\psi_2(a \oplus b^\circ) = (J_\pi \pi(b)^* J_\pi \xi, \xi)P_h = (\pi(b)\xi, \xi)P_h.$$

Noting $A = \mathcal{A}_{sa} = \{a \oplus a^\circ; a \in A\} \subset C^*(A)$ (cf. [44; Cor.4.5]), we have

$$\psi_1(a \oplus a^\circ) = \psi_2(a \oplus a^\circ) = (\pi(a)\xi, \xi)P_h = \omega_\pi(a)P_h.$$

i.e.,

$$\psi_1|_A = V^*\pi V = V^*\pi^t V = \psi_2|_A \qquad \text{while} \qquad \psi_1 \neq \psi_2.$$

In the above example, the state $\omega_\pi P_h \in Q_H(A)$ extends to two different CP-maps in its enveloping C*-algebra $C^*(A)$, i.e., $\psi_1 \in Q_H(\mathcal{A})$ and $\psi_2 \in Q_H^t(\mathcal{A}) = Q_H(\mathcal{A}^\circ)$, each of which corresponds to the choice of orientation on A, i.e., $\mathcal{A}$ or $\mathcal{A}^\circ$. We are now in the position to realize that the CP-state space can throw new sight on the orientability condition.

DEFINITION 7.6. *Let A be a JC-algebra and H be a Hilbert space. Then, for each $\psi = V^*\kappa V \in Q_H(C^*(A))$, we define the **transposed** CP-state $\psi^T \in Q_H(C^*(A))$ of ψ by*

$$\psi^T := V^*(\kappa^t \circ \Phi)V = \psi^t \circ \Phi$$

*where Φ is the canonical *-anti-automorphism of $C^*(A)$. We define $\psi \in Q_H(C^*(A))$ to be **symmetric** if $\psi^T = \psi$. We denote by $Q_H^s(C^*(A))$ the set of all symmetric CP-states in $Q_H(C^*(A))$. Similarly, we can define $Q_H^s(W^*(M))_n$ for a JW-algebra M.*

Remark. Note that $\psi^T|_A = \psi^t$ since A is Φ-invariant.

In order to understand the meaning implied in the above definition, let us find the expression of the symmetric elements in the particular case that $A = \mathcal{A}_{sa}$ for a C*-algebra $\mathcal{A}$ with no one dimensional representation (i.e., $C^*(A) = \mathcal{A} \oplus \mathcal{A}^\circ$). Let $\psi = V^*\kappa V \in Q_H(C^*(A))$. As we saw in the proof of Theorem 7.4, we can set $\kappa = \pi_1 \oplus \pi_2$ with *-representations π_i $(i = 1, 2)$ of $\mathcal{A}$ on H_{π_i}, and $V = V_1 \oplus V_2$ with $V_i \in B(H, H_{\pi_i})$. Then, for all $a \oplus b^\circ \in \mathcal{A} \oplus \mathcal{A}^\circ = C^*(A)$,

$$\psi(a \oplus b^\circ) = V_1\pi_1(a)V_1 + V_2\pi_2^t(b)V_2 = \varphi_1(a) + \varphi_2^t(b),$$

where we set $\varphi_i = V_i^*\pi_i V_i \in Q_H(\mathcal{A})$ $(i = 1, 2)$. Recalling $\Phi(a \oplus b^\circ) = b \oplus a^\circ$ (e.g., [44]), we have

$$\begin{aligned}\psi^T(a \oplus b^\circ) &= (V_1 \oplus V_2)^*(\pi_1^t \oplus \pi_2^t)(b \oplus a^\circ)(V_1 \oplus V_2)\\ &= V_1^*\pi_1^t(b)V_1 + V_2^*\pi_2(a)V_2\\ &= \varphi_1^t(b) + \varphi_2(a)\end{aligned}$$

It follows from this that $\psi^T = \psi$ if and only if $\varphi_1 = \varphi_2$, i.e., every symmetric element $\psi \in Q_H^s(C^*(A))$ is of the form

$$\psi = V^*\pi V + V^*\pi^t V$$

where π is a *-representation of $C^*(A)$ on H_π and $V \in B(H, H_\pi)$ with $2V^*V \leq I_H$, or simply we can write

$$\psi = \frac{1}{2}(\varphi + \varphi^t) \qquad \text{with} \quad \varphi \in Q_H(\mathcal{A}).$$

DEFINITION 7.7. *Let A be a JC-algebra and H be a Hilbert space, and let $\psi_i = V_i^*\rho_i V_i \in Q_H(A)(i = 1, 2)$. Then, we say that ψ_1 and ψ_2 are* ***transposed*** *equivalent* if $\psi_1 = \psi_2^t$, and write $\psi_1 \overset{t}{\sim} \psi_2$. *We denote by* $\tilde{Q}_H(A)$ *the set of all transpose equivalence classes of* $Q_H(A)$, *and call* $\tilde{Q}_H(A)$ *the reduced CP-state space of* A. *Similarly, for a JW-algebra* M, *we define the reduced normal CP-state space of* M *and denote it by* $\tilde{Q}_H(M)_n$.

PROPOSITION 7.8.A. *Let* A *be a universally reversible JC-algebra, and* H *be a Hilbert space. Then,* $Q_H^s(C^*(A))$ *is mapped bijectively onto* $\tilde{Q}_H(A)$ *by the restriction map.*

PROPOSITION 7.8.B. *Let* M *be a universally reversible JW-algebra, and* H *be a Hilbert space. Then,* $Q_H^s(W^*(M))_n$ *is mapped bijectively onto* $\tilde{Q}_H(M)_n$ *by the restriction map.*

PROOF: It is enough to prove Part B since $Q_H^s(C^*(A)) = Q_H^s(W^*(A^{**}))_n$. Let $\psi_1, \psi_2 \in Q_H^s(W^*(M))$ and assume $r(\psi_1) \sim r(\psi_2)$, i.e., $\psi_1 = \psi_1^T = \psi_1^t \circ \Phi$, $\psi_2 = \psi_2^T = \psi_2^t \circ \Phi$, and $\psi_1(a) = \psi_2(a)$ or $\psi_1(a) = \psi_2^t(a)$ for all $a \in M$. Note that, for any $x \in W^*(M)_{sa}$, $\frac{1}{2}(x + \Phi(x)) \in W^*(M)_{sa}^{\Phi} = M$ (see Proof of Lemma 7.1.B), so

$$\psi_1(\frac{1}{2}(x + \Phi(x))) = \psi_2(\frac{1}{2}(x + \Phi(x))) \quad \text{or} \quad \psi_1(\frac{1}{2}(x + \Phi(x))) = \psi_2^t(\frac{1}{2}(x + \Phi(x))),$$

i.e.,

$$\frac{1}{2}(\psi_1(x) + (\psi_1 \circ \Phi)(x)) = \frac{1}{2}(\psi_2(x) + (\psi_2 \circ \Phi)(x))$$

or

$$\frac{1}{2}(\psi_1(x) + (\psi_1 \circ \Phi)(x)) = \frac{1}{2}(\psi_2^t(x) + (\psi_2^t \circ \Phi)(x)).$$

Substituting $\psi_i \circ \Phi = \psi_i^t$ and $\psi_i^t \circ \Phi = \psi_i$ $(i = 1, 2)$, we have in the both cases

$$\psi_1(x) + \psi_1^t(x) = \psi_2(x) + \psi_2^t(x) \qquad \text{for all } x \in W^*(M)_{sa},$$

so that

$$\psi_1 + \psi_1^t = \psi_2 + \psi_2^t.$$

It follows from this that $\psi_1 - \psi_2 = -(\psi_1 - \psi_2)^t$, so

$$\psi_1 - \psi_2 \in CB(W^*(M), B(H))_n \cap CB^t(W^*(M), B(H))_n = CB(W^*(M), B(H))_n^{ab},$$

so that $(\psi_1 - \psi_2)^t = \psi_1 - \psi_2$. Hence, we have $\psi_1 - \psi_2 = -(\psi_1 - \psi_2)$, i.e., $\psi_1 = \psi_2$. Thus, r is a one to one map from $Q_H^s(W^*(M))_n$ to $\tilde{Q}_H(M)_n$.

To see that it is onto, let $\psi \in Q_H(M)_n$ and define $\bar{\psi}^s := \frac{1}{2}(\bar{\psi} + \bar{\psi}^T)$ where $\bar{\psi}$ is the canonical extension of ψ. Clearly, $\bar{\psi}^s \in Q_H^s(W^*(M))_n$, and

$$r(\bar{\psi}^s) = \frac{1}{2}(r(\bar{\psi}) + r(\bar{\psi}^T)) = \frac{1}{2}(r(\bar{\psi}) + r(\bar{\psi}^t \circ \Phi)) = \frac{1}{2}(\psi + \psi^t) \sim \psi.$$

Hence, $r(Q_H^s(W^*(M))_n) = \tilde{Q}_H(M)_n$. ∎

Note that the reduced CP-state space $\tilde{Q}_H(A)$ is a CP-convex set, since $Q_H^s(C^*(A))$ is CP-convex and $\tilde{Q}_H(A) = r(Q_H^s(C^*(A)))$.

Proposition 7.9.A. *Let A be a JC-algebra and H be a Hilbert space with $dim H \geq \alpha_c(A)$. Then,*

$$A \cong AC(\tilde{Q}_H(A), B(H)_{sa}) \qquad \textit{(Jordan isomorphism)}$$

Proposition 7.9.B. *Let M be a JW-algebra and H be a Hilbert space with $dim H \geq \alpha_c(M)_n$. Then*

$$M \cong AB(\tilde{Q}_H(M)_n, B(H)_{sa}) \qquad \textit{(Jordan isomorphism)}$$

Proof: It is sufficient to prove Part B, since Part A is obtained from Part B by taking $M = A^{**}$ and considering the BW-w continuous elements.

We consider the following diagram;

$$\begin{tikzcd}
W^*(M) \arrow[r, "\cong"] & AB(Q_H(W^*(M))_n, B(H))_{sa} \\
M \arrow[u] \arrow[r, "\tilde{i}"] \arrow[dr, "i"'] & AB(\tilde{Q}_H(M)_n, B(H)_{sa}) \arrow[u, "(s\circ r)^\natural"'] \arrow[d, "s^\natural"] \\
 & AB(Q_H(M)_n, B(H)_{sa})
\end{tikzcd}$$

In the above commutative diagram, i and $\tilde{i}$ denote the evaluation maps, and $(s\circ r)^\natural$ and $s^\natural$ are the CP-dual maps of the restriction map $(s\circ r) : Q_H(W^*(M))_n \to \tilde{Q}_H(M)_n$ and the canonical quotient map $s : Q_H(M)_n \to \tilde{Q}_H(M)_n$ respectively.

It is straightforward to see that $\tilde{i}$ is a injective embedding. In fact, for $a \in M$, if $\psi(a) = 0$ for all $\psi \in \tilde{Q}_H(M)_n$, then $\psi(a) = \psi^t(a) = 0$ for all $\psi \in Q_H(M)_n$, which implies $a = 0$.

To see that $\tilde{i}$ is surjective, we need to show $s^\natural$ is injective. Let $\gamma \in AB(\tilde{Q}_H(M)_n, B(H)_{sa})$ and assume $s^\natural(\gamma)(\psi) = \gamma(s(\psi)) = 0$ for all $\psi \in Q_H(M)_n$. By Proposition 7.8, $s(\psi) = (s\circ r)(\bar{\psi}^s) = (s\circ r)(\frac{1}{2}(\bar{\psi} + \bar{\psi}^T))$ where $\bar{\psi}$ denotes the canonical extension of ψ. So, for all $\psi \in Q_H(M)_n$,

$$\begin{aligned}
\gamma(s(\psi)) = \gamma((s\circ r)(\frac{1}{2}(\bar{\psi} + \bar{\psi}^T))) &= (s\circ r)^\natural(\gamma)(\frac{1}{2}(\bar{\psi} + \bar{\psi}^T)) \\
&= \frac{1}{2}\{(s\circ r)^\natural(\gamma)(\bar{\psi}) + (s\circ r)^\natural(\gamma)(\bar{\psi}^T)\} = 0.
\end{aligned}$$

It is enoughh to prove the case $\gamma \geq 0$, for which we have $(s \circ r)^{\natural}(\gamma) \geq 0$ and $(s \circ r)^{\natural}(\gamma)(\bar{\psi}) = (s \circ r)^{\natural}(\gamma)(\bar{\psi}^T) = 0$. Since $\bar{\psi}$ moves all over $Q_H(W^*(M))_n$, from the CP-duality of $W^*(M)$, we have $(s \circ r)^{\natural}(\gamma) = 0$. Note that $(s \circ r)^{\natural}$ is one to one since $s \circ r$ is onto, so that we conclude $\gamma = 0$. This proves the isomorphism. ∎

We can now reformulate the CP-orientability condition obtained in Theorem 7.4.

THEOREM 7.10.A. *Let A be a universally reversible JC-algebra and H be a Hilbert sapce with $dim H \geq \alpha_c(A)$. Then, A is the self-adjoint part of some C^*-algebra $\mathcal{A}$ if and only if. for any $\psi \in \tilde{Q}_H(A)$, there exist a *-representation $\pi : \mathcal{A} \to B(H_\pi)$ and $V \in B(H, H_\pi)$ with $V^*V \leq I_H$, such that the symmetric extension $\bar{\psi}^s$ of ψ is expressed uniquely as*

$$\bar{\psi}^s = \frac{1}{2}(V^*\pi V + V^*\pi^t V),$$

i.e.,

$$\bar{\psi}^s = \frac{1}{2}(\varphi + \varphi^t) \qquad \textit{with} \quad \varphi \in Q_H(\mathcal{A}).$$

THEOREM 7.10.B. *Let M be a universally reversible JW-algebra and H be a Hilbert space with $dim H \geq \alpha_c(M)_n$. Then, M is the self-adjoint part of some W^*-algebra $\mathcal{M}$ if and only if, for any $\psi \in \tilde{Q}_H(M)_n$, there exist a normal *-representation $\pi : M \to B(H_\pi)$ and $V \in B(H, H_\pi)$ with $V^*V \leq I_H$, such that the symmetric extension $\bar{\psi}^s$ of ψ is expressed uniquely as*

$$\bar{\psi}^s = \frac{1}{2}(V^*\pi V + V^*\pi^t V),$$

i.e.,

$$\bar{\psi}^t = \frac{1}{2}(\varphi + \varphi^t) \qquad \textit{with} \quad \varphi \in Q_H(\mathcal{M})_n.$$

PROOF: We prove Part B first. Assume that M is the self-adjoint part of a W*-algebra $\mathcal{M}$, and let $\psi \in \tilde{Q}_H(M)_n$. By Proposition 7.8.B, ψ is uniquely extended to $\bar{\psi}^s \in Q_H^s(W^*(M))_n$, and then by Corollary 7.5.B, $\bar{\psi}^s$ is decomposed as

$$\bar{\psi}^s = V_1^*\pi_1 V_1 + V_2^*\pi_2^t V_2 + V_0^*\pi_0 V_0$$

where π_1, π_2 are non-abelian normal representations and π_0 is an abelian normal representation, and $V_i \in B(H, H_{\pi_i})(i = 0, 1, 2)$ with $\sum_{i=0}^2 V_i^* V_i \leq I_H$. We will show that the symmetry $\bar{\psi}^s = (\bar{\psi}^s)^T$ requires $V_1^*\pi_1 V_1 = V_2^*\pi_2 V_2$. In fact, noting $\Phi(a \oplus b^\circ \oplus c) = b \oplus a^\circ \oplus c$ for $a \oplus b^\circ \oplus c \in W^*(M) = \mathcal{M}_{nab} \oplus \mathcal{M}_{nab}^\circ \oplus \mathcal{M}_{ab}$, and recalling the proof of Theorem 7.4,

$$\begin{aligned}
(\bar{\psi}^s)^T(a \oplus b^\circ \oplus c) &= (\bar{\psi}^s)^t \circ \Phi(a \oplus b^\circ \oplus c) \\
&= (V_1 \oplus V_2 \oplus V_0)^*(\pi_1^t \oplus \pi_2^t \oplus \pi_0^t)(b \oplus a^\circ \oplus c)(V_1 \oplus V_2 \oplus V_0) \\
&= V_1^*\pi_1^t(b)V_1 + V_2^*\pi_2(a)V_2 + V_0^*\pi_0(c)V_0.
\end{aligned}$$

Thus $\bar{\psi}^s = (\bar{\psi}^s)^T$ implies $V_1\pi_1V_1 = V_2\pi_2V_2$. We set here $\varphi = V^*\pi V$ where $\pi = \pi_1 \oplus \pi_0$ and $V = \sqrt{2}V_1 \oplus V_0$, then $\pi : \mathcal{M} \to B(H_\pi)$ is a normal *-representation of $\mathcal{M}$ and $V \in B(H, H_\pi)$ with $V^*V = 2V_1^*V_1 + V_0^*V_0 = \sum_{i=0}^2 V_i^*V_i \leq I_H$ satisfying $\varphi = 2V_1^*\pi_1V_1 + V_0^*\pi_0V_0 = 2V_2^*\pi_2V_2 + V_0^*\pi_0V_0 \in Q_H(\mathcal{M})_n$. Hence, we have a simpler expression

$$\bar{\psi}^s = \frac{1}{2}(\varphi + \varphi^t) \qquad \text{with} \qquad \varphi \in Q_H(\mathcal{M})_n.$$

Conversely, assume the unique expression of $Q_H^s(W^*(M))_n$ claimed in the theorem. Let $\psi \in Q_H(M)_n$. Then, by assumption, $\bar{\psi}^s = \frac{1}{2}(\varphi + \varphi^t)$ where $\varphi \in Q_H(\mathcal{M})_n$. It follows from this that there exists a bijection by the restriction map

$$r : \varphi \in Q_H(\mathcal{M})_n \;\; \mapsto \;\; \frac{1}{2}(\varphi + \varphi^t) \in \tilde{Q}_H(M)_n,$$

which is a CP-affine isomorphism, and further it is an isometry, since

$$\|\varphi\| = \|\varphi(e)\| = \|\frac{1}{2}(\varphi(e) + \varphi^t(e))\| = \|\frac{1}{2}(\varphi + \varphi^t)\|.$$

Hence, by Proposition 7.9.B and Theorem 3.7.B, we have

$$\begin{aligned} M &\cong AB(\tilde{Q}_H(M)_n, B(H)_{sa}) \\ &\cong AB(Q_H(\mathcal{M})_n, B(H)_{sa}) \\ &\cong \mathcal{M}_{sa}. \end{aligned}$$

Part A follows from Part B since A is the self-sdjoint part of some C*-algebra $\mathcal{A}$ if and only if A^{**} is the self-adjoint part of $\mathcal{A}^{**}$, and note that $Q_H(A) = Q_H(A^{**})_n$ and $Q_H(C^*(A)) = Q_H(W^*(A^{**}))_n$. ∎

COROLLARY 7.11.A. *Under the condition of Theorem 7.10.A, A is the self-adjoint part of a C*-algebra $\mathcal{A}$ if and only if $\tilde{Q}_H(A) \underset{\text{CP}}{\cong} Q_H(\mathcal{A})$ (CP-affine BW-homeomorphism).*

CORROLLARY 7.11.B. *Under the condition of Theorem 7.10.B, M is the self-adjoint part of a W*-algebra $\mathcal{M}$ if and only if $\tilde{Q}_H(M)_n \underset{\text{CP}}{\cong} Q_H(\mathcal{M})_n$ (CP-affine isomorphism).*

PROOF: See the proof of Theorem 7.10. For Part A, note that

$$r : \varphi \in Q_H(\mathcal{A}) \mapsto \frac{1}{2}(\varphi + \varphi^t) \in \tilde{Q}_H(A)$$

is a CP-affine BW-homeomorphism since r is a one to one BW-continuous map from the BW-compact set $Q_H(\mathcal{A})$ onto $\tilde{Q}_H(A)$ which is Hausdorff in the quotient topology. ∎

It will be convenient to restate the orientability conditions which we obtained so far.

THEOREM 7.12.A. *Let A be a universally reversible JC-algebra and H be a Hilbert space with $dim H \geq \alpha_c(A)$. Then, A is the self-adjoint part of a C^*-algebra $\mathcal{A}$ if and only if one of the following equivalent conditions is satisfied.*

(i) $Q_H(C^*(A)) = Q_H(\mathcal{A})_{nab} \oplus_{CP} Q^t_H(\mathcal{A})_{nab} \oplus_{CP} Q_H(\mathcal{A})_{ab}$

(ii) For any $\psi \in \tilde{Q}_H(A)$, there exists unique $\varphi \in Q_H(\mathcal{A})$ such that

$$\bar{\psi}^s = \tfrac{1}{2}(\varphi + \varphi^t).$$

(iii) $\tilde{Q}_H(A) \underset{CP}{\cong} Q_H(\mathcal{A})$ *(CP-affine BW-homeomorphism)*

THEOREM 7.12.B. *Let M be a universally reversible JW-algebra and H be a Hilbert space with $dim H \geq \alpha_c(M)_n$. Then, M is the self-adjoint part of a W^*-algebra $\mathcal{M}$ if and only if one of the following conditions is satisfied.*

(i) $Q_H(W^*(M))_n = Q_H(\mathcal{M})^{nab}_n \oplus_{CP} Q^t_H(\mathcal{M})^{nab}_n \oplus_{CP} Q_H(\mathcal{M})^{ab}_n$

(ii) For any $\psi \in \tilde{Q}_H(\mathcal{M})_n$, there exists unique $\varphi \in Q_H(\mathcal{M})_n$ such that

$$\bar{\psi}^s = \tfrac{1}{2}(\varphi + \varphi^t).$$

(iii) $\tilde{Q}_H(M)_n \underset{\mathrm{CP}}{\cong} Q_H(\mathcal{M})$ *(CP-affine isomorphism)*

We will call $\tilde{Q}_H(A)$ (or $\tilde{Q}_H(M)_n$) to be *CP-orientable* if one of the above equivalent conditions is satisfied. Note that the CP-orientability conditions are symmetric with respect to the initiative of $Q_H(\mathcal{A})_{nab}$ or $Q^t_H(\mathcal{A})_{nab}$, e.g., the condition Theorem 7.12.A.(iii) could be stated $\tilde{Q}_H(A) \underset{\mathrm{CP}}{\cong} Q^t_H(\mathcal{A})$. Hence, if $\tilde{Q}_H(\mathcal{A})$ is CP-orientable, we can give it an orientation which corresponds to $\mathcal{A}$ or $\mathcal{A}^\circ$.

PROPOSITION 7.13.A. *Let A be a universally reversible JC-algebra and H be a Hilbert space with $dim H \geq \alpha_c(A)$, and assume that $\tilde{Q}_H(A)$ is CP-orientable. Then, the orientation is defined on each open and closed CP-split face of $\tilde{Q}_H(A)$.*

PROPOSITION 7.13.B. *Let M be a universally reversible JW-algebra and H be a Hilbert space with $dim H \geq \alpha_c(M)_n$, and assume that $\tilde{Q}_H(M)_n$ is CP-orientable. Then, the orientation is defined on each CP-split face of $\tilde{Q}_H(M)_n$.*

PROOF: Part B: By the CP-duality between $W^*(M)$ and $Q_H(W^*(M))_n$, the CP-direct decomposition of $Q_H(W^*(M))_n$ corresponds to the algebraic decomposition of $W^*(M)$, and the orientability conditions given in Theorem 7.12.B are all independent to each CP-split face of $Q_H(W^*(M))_n$, hence of $\tilde{Q}^s_H(W^*(M))_n(\cong \tilde{Q}_H(M)_n)$. This proves our claim for W*-algebra case.

Part A: The same argument is not applied for C*-algebra case in general. However, if there exists an open and closed central projection $p \in A$, then by Proposition 4.16, A is decomposed into the algebraic direct sum $A = U_p \oplus U_{e-p} = pAp \oplus (e-p)A(e-p)$, and the orientation is defined independently on each direct summand from the previous argument for the W*-algebra case. ∎

Remark. P.J. Stacey [70] showed that the state space of a JBW-algebra of complex type (see Appendix 3 for definition) is orientable if and only if it is *locally* orientable, i.e., for each pure state ω, the closed split face generated by ω is orientable. Our result (Proposition 7.13.B) claims just the "normal part" of the local orientability.

We now study the relation between our CP-orientability condition on the CP-state spaces and the 3-ball orientability condition for the state spaces of JB-algebras of complex type (see Appendix 3 for definition) studied in [4], [6], [47], [68], [69]. We recall here that a JB-algebra A of complex type is characterized by the *3-ball property* ([4]), i.e., for any pure states $\sigma, \tau \in P(A)$, the face($\{\sigma, \tau\}$) generated by σ and τ is affinely isomorphic to a 3-dimensional ball if $\sigma \simeq \tau$, and isomorphic to the line segment $[\sigma, \tau]$ if $\sigma \not\simeq \tau$. It is shown in [4] that the self-adjoint part of a C*-algebra is of complex type, and that every JC-algebra of complex type is universally reversible.

At first, it should be noted that the setting of these approaches have subtle difference, i.e., our CP-orientability conditions are stated in the general setting of universally reversible JC-algebras, which enabled us to formulate our orientability conditions without reference to 3-ball property and in parallel for both C*- and W*-algebras, while the orientability condition for the normal state space of JBW-algebras of complex type, in the context of 3-ball property discussed in [47], turned out to be very elabolate, since there may not be pure states in the normal state space of JBW-algebras.

In the following, we will show that there exists a close relationship between 3-ball orientability and CP-orientability if we restrict our attention to JB-algebras of complex type.

We recall from [4] that the state space $S(A)$ of a JB-algebrea of complex type is said to be *orientable* if the $\mathbb{Z}_2$ bundle $A(E^3, S(A))/SO(3) \to A(E^3, S(A))/O(3)$ is trivial, where $A(E^3, S(A))$ denotes the set of all affine maps from 3-dimensional ball E^3 onto 3-balls on $S(A)$ with the topology of point wise convergence and groups $SO(3)$ and $O(3)$ are acting continuously on $A(E^3, S(A))$ as composition. It is shown in [68; Lemma 2] that the dual map of a *-preserving Jordan homomorphism ϕ between JB-algebras A_i $(i = 1, 2)$ of complex type maps 3-balls onto 3-balls. Then, $\phi^* : S(A_1) \to S(A_2)$ is said to be *orientation preserving* if it preserves the orientation of each 3-ball, i.e., if $\phi^*(B_1) = B_2$ with 3-balls B_i $(i = 1, 2)$, $det(\theta_2^{-1} \circ \phi^* \circ \theta_1) = 1$ for any parametrization $\theta_i : E_3 \to B_i$ $(i = 1, 2)$ in the given orientation defined from the GNS-representation associated with the pure states of B_i. (see [68], [69] for details). We will show directly the relation between the orientation preserving maps and CP-affine maps as follows.

PROPOSITION 7.14. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital C^*-algebras and let H be a Hilbert space with $dimH \geq \sup_{i=1,2} \alpha_c(\mathcal{A}_i)$. If $\Psi : Q_H(\mathcal{A}_1) \to Q_H(\mathcal{A}_2)$ is a CP-affine isomorphism, then $\Psi|_{S(\mathcal{A}_1)} : S(\mathcal{A}_1) \to S(\mathcal{A}_2)$ preserves 3-balls and their orientation. Conversely, if $\Psi_0 : S(\mathcal{A}_1) \to S(\mathcal{A}_2)$ is an affine isomorphism which preserves 3-balls and their orientation, then it induces a CP-affine isomorphism $\Psi : Q_H(\mathcal{A}_1) \to Q_H(\mathcal{A}_2)$.*

PROOF: Let $\sigma, \tau \in P(\mathcal{A}_1)$ with $\sigma \simeq \tau$. We recall that σ and τ can be expressed as one dimensional CP-states, i.e.,

$$\psi_\sigma = PU_\sigma^* \pi_\sigma U_\sigma P \qquad \text{and} \qquad \psi_\tau = PU_\tau^* \pi_\tau U_\tau P$$

where π_σ and π_τ are the GNS-representations associated with σ and τ respectively, and $U_\sigma : H \to H_\sigma$, $U_\tau : H \to H_\tau$ are onto partial isometries, and P is a one dimensional projection on H (cf. Proof of Theorem 2.2).

It is straightforward to see that $\Psi(\sigma), \Psi(\tau) \in P(\mathcal{A}_2)$; since Ψ is a CP-affine isomorphism, it maps one dimensional pure rays of $Q_H(\mathcal{A}_1)$ $(= P(\mathcal{A}_1))$ onto one dimensional pure rays of $Q_H(\mathcal{A}_2)$ $(= P(\mathcal{A}_2))$. We also note that $\Psi(\sigma) \simeq \Psi(\tau)$. In fact, since $\sigma \simeq \tau$, there exists a unitary $u \in \mathcal{A}_1$ such that

$$U_\tau^* \pi_\tau U_\tau = U_\tau^* \pi_\sigma(u^*(\cdot)u) U_\tau = U_\tau^* \pi_\sigma(u)^* U_\sigma (U_\sigma^* \pi_\sigma U_\sigma) U_\sigma^* \pi_\sigma(u) U_\tau$$

(cf. [28; Cor. 2.8.6]) so that

$$\Psi(U_\tau^* \pi_\tau U_\tau) = (U_\sigma^* \pi_\sigma(u) U_\tau)^* \Psi(U_\sigma^* \pi_\sigma U_\sigma)(U_{\sigma^*} \pi_\sigma(u) U_\tau),$$

i.e.,

$$\pi_{\Psi(\tau)} \simeq \Psi(U_\tau^* \pi_\tau U_\tau) \simeq \Psi(U_\sigma^* \pi_\sigma U_\sigma) \simeq \pi_{\Psi(\sigma)},$$

which implies $\Psi(\sigma) \simeq \Psi(\tau)$.

Clearly, $\Psi|_{S(\mathcal{A}_1)} : S(\mathcal{A}_1) \to S(\mathcal{A}_2)$ is an affine isomorphism, so that it induces a *-preserving Jordan isomomorphism (e.g., [19; Th.3.2.3]). Hence, by [68; Lemma 2], Ψ maps the 3-ball $B(\sigma, \tau)$ onto the 3-ball $B(\Psi(\sigma), \Psi(\tau))$. We have to show Ψ preserves the orientations of these 3-balls.

Let $p_1 \in \mathcal{P}(\mathcal{A}_1^{**})$, $p_2 \in \mathcal{P}(\mathcal{A}_2^{**})$ be the projections corresponding to the projective faces $B(\sigma,\tau)$ and $B(\Psi(\sigma),\Psi(\tau))$ respectively. We will show that the CP-dual map

$$\Psi^{\natural} : U_{p_2}(\mathcal{A}_2^{**}) = p_2\mathcal{A}_2^{**}p_2 \quad \longrightarrow \quad U_{p_1}(\mathcal{A}_1^{**}) = p_1\mathcal{A}_1^{**}p_1$$

is 2-positive. Then, Choi's result [23; Cor. 3.2] claims that every 2-positive *-Jordan isomorphism is a *-isomorphism, i.e., it preserves orientation, which completes our proof.

To see this, let $\gamma_{ij} \in p_2\mathcal{A}_2^{**}p_2 \cong AB(F_H(p_2), B(H))(i,j = 1,2)$ (cf. Theorem 4.9), and consider the induced map

$$\Psi_2^{\natural} : \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \mapsto \begin{pmatrix} \Psi^{\natural}(\gamma_{11}) & \Psi^{\natural}(\gamma_{12}) \\ \Psi^{\natural}(\gamma_{21}) & \Psi^{\natural}(\gamma_{22}) \end{pmatrix}.$$

Assume that

$$\sum_{ij}(\gamma_{i,j}(\varphi)\xi_i,\xi_j) \geq 0 \qquad \text{for all} \quad \varphi \in F_H(p_2) \text{ and } \xi_i,\xi_j \in H \ \ (i,j = 1,2),$$

then, for any $\psi \in F_H(p_1)$, $\Psi(\psi) \in F_H(p_2)$, so that

$$\sum_{i,j}(\Psi^{\natural}(\gamma_{ij})(\psi)\xi_i,\xi_j) = \sum_{i,j}(\gamma_{ij}(\Psi(\psi))\xi_i,\xi_j) \geq 0.$$

This shows that $\Psi^{\natural}$ is 2-positive.

Conversely, suppose that $\Psi_0 : S(\mathcal{A}_1) \to S(\mathcal{A}_2)$ is an affine isomorphism which preserves 3-balls and their orientation. Then, by [19; Th. 3.2.3], $\Psi_0^* : \mathcal{A}_2 \to \mathcal{A}_1$ is a *-preserving Jordan isomorphism, and under this condition, [68; Prop.3] shows that, for each irreducible representation $\pi \in Irr(\mathcal{A}_1)$, $\pi \circ \Psi_0^*$ is an irreducible representation of $\mathcal{A}_2$, i.e., $\pi \circ \Psi_0^* \in Irr(\mathcal{A}_2)$. This shows that the CP-dual of Ψ_0^*, which we denote by Ψ, induces a non-flip CP-affine isomorphism $\Psi : Q_H(\mathcal{A}_1) \cong Q_H(\mathcal{A}_2)$. This completes the proof. ▮

CHAPTER 8

CP-GEOMETRIC REALIZATION OF TOMITA-TAKESAKI THEORY

In this chapter, we study the standard structure of W*-algebras in our setting of CP-convexity. We shall first show that every W*-algebra has a faithful normal CP-state which is shown to be the counterpart of a faithful normal semi-finite weight in the usual theory, and we define the standard representation by "CP-GNS construction". By letting the opposite algebra play the role of the non-spatial commutant, we generalize the standard structure in non-standard normal representations, and realize Tomita-Takesaki's anti-isomorphism theorem in the setting of CP-duality; this was not realized when we stayed in the scalar convexity.

We shall next study the hyperstandard structure of W*-algebras. We focus on Connes's approach to orientability condition through the derivations of a homogeneous self-dual cone, which is the abstract generalization of the natural self-dual cone of a standard von Neumann algebra, and clarify its relation with the CP-orientability condition obtained in Chapter 7. Our arguments are based on the CP-duality of the enveloping W*-algebra of the self-adjoint derivations, which has the structure of real Jordan and Lie algebra, where we can see interesting overlaps between the Tomita-Takesaki theory and the structure theory of JBW-algebras. This approach also enables us to represent the bounded *-derivations and

*-automorphisms in our CP-geometrical setting. We shall finally discuss Tomita-Takesaki's one-parameter modular group for JW-algebras. (The σ-finite case has already been obtained by U. Haagerup and H. Hanche-Olsen [42]).

DEFINITION 8.1. *Let $\mathcal{A}$ be a C^*-algebra and let H be a Hilbert space. A CP-state $\psi \in Q_H(\mathcal{A})$ is defined to be **faithful** if $\psi(a) = 0$ for $a \in \mathcal{A}^+$ implies that $a = 0$.*

*Let $\mathcal{M}$ be a W^*-algebra. For each $\psi \in Q_H(\mathcal{M})_n$, we define the **support projection** $s(\psi)$ of ψ by*

$$s(\psi) := \wedge\{p \in \mathcal{P}(\mathcal{M}) : \psi \in F_H(p)\}.$$

*For $\psi, \varphi \in Q_H(\mathcal{M})_n$, we write $\psi \perp \varphi$ if $s(\psi) \perp s(\varphi)$ and we say that ψ and φ are mutually orthogonal. Relatedly, we define the **support projective face** $F_H[\psi]$ of ψ by*

$$F_H[\psi] := \wedge\{F_H(p) : \psi \in F_H(p)\} \ (= F_H(s(\psi))).$$

LEMMA 8.2. *Let $\mathcal{M}$ be a W^*-algebra and H be a Hilbert space. Then the following conditions are equivalent.*

(i) $\psi \in Q_H(\mathcal{M})_n$ *is a faithful normal CP-state.*

(ii) $s(\psi) = e$

(iii) $F_H[\psi] = Q_H(\mathcal{M})_n$

PROOF: (i) $\Leftrightarrow$ (ii) : $({}^{\perp}\psi)^+ = \{a \in \mathcal{M}^+ : \psi(a) = 0\}$ is a w*-closed hereditary subcone of $\mathcal{M}^+$. Let $p_0 \in \mathcal{M}$ be the support projection of the order ideal generated by $({}^{\perp}\psi)^{+}$. Then,

$$\begin{aligned} p_0 &= \vee\{p \in \mathcal{P}(\mathcal{M}) : \psi(p) = 0\} \\ &= e - \wedge\{q = e - p \in \mathcal{P}(\mathcal{M}) : \psi(p) = 0\} \\ &= e - \wedge\{q \in \mathcal{P}(\mathcal{M}) : \psi \in F_H(q)\} \\ &= e - s(\psi) \end{aligned}$$

Hence, $({}^{\perp}\psi)^{+} = p_0\mathcal{M}^{+}p_0 = 0$ if and only if $s(\psi) = e$.

(ii) $\Leftrightarrow$ (iii) : This is immediate from the definition. ∎

DEFINITION 8.3. *Let $\mathcal{M}$ be a W^*-algebra. We denote by $\alpha_\sigma(\mathcal{M})$ the maximal cardinality of the mutually orthogonal projections in $\mathcal{M}$.*

Remark. We note that, if $\mathcal{M}$ is σ-finite, then $\alpha_\sigma(\mathcal{M}) \leq \alpha_c(\mathcal{M})$. In fact, in this case $\mathcal{M}$ has a faithful normal state ω_0, and $\mathcal{M}$ is represented standardly by the GNS-representation π_{ω_0} associated with ω_0. Then, a family of mutually orthogonal projections $(e_i)_{i\in I}$ in $\mathcal{M}$ forms an orthogonal set in H_{ω_0} since $\omega_0(e_i^* e_j) = 0$ for $i \neq j$ and $\omega_0(e_i) \neq 0$ for all $i \in I$, and note that π_{ω_0} is a cyclic representation. $\alpha_\sigma(\mathcal{M})$ is also defined as the cardinality of maximal family of mutually orthogonal normal CP-states.

PROPOSITION 8.4. *Let $\mathcal{M}$ be a W^*-algebra and let H be a Hilbert space with $dim H \geq \alpha_\sigma(\mathcal{M})$. Then, $\mathcal{M}$ has a faithful normal CP-state in $Q_H(\mathcal{M})_n$.*

PROOF: Let $(\pi_i)_{i\in I}$ be a maximal family of normal cyclic representations whose supports $s(\pi_i)$ are mutually orthogonal, and for each $i \in I$, let ξ_i be a cyclic vector of π_i with $\|\xi_i\| \leq 1$. We then take $V_i \in B(H, H_{\pi_i})$ such that $\xi_i \in V_i H$ and $\sum_{i\in I} V_i^* V_i \leq I_H$.

For instance, we can set

$$V_i h = (h, e_i)\xi_i \qquad \text{for } h \in H \text{ and } i \in I$$

where $(e_i)_{i\in I}$ is an orthonormal set in H. Then, $V_i e_i = \xi_i$ and, for any $h \in H$,

$$((\sum_{i\in I} V_i^* V_i)h, h) = \sum_{i\in I}(V_i h, V_i h) = \sum_{i\in I} |(h, e_i)|^2 \|\xi_i\|^2 \leq (h, h),$$

so that we have indeed $\sum_{i\in I} V_i^* V_i \leq I_H$.

We set $\pi = \oplus_{i\in I}\pi_i$ and $V = \oplus_{i\in I} V_i$ and define $\psi_0 = V^*\pi V = \sum_{i\in I} V_i^* \pi_i V_i$. Then, $\|\psi_0\| = \|\psi_0(e)\| = \|\sum_{i\in I} V_i^* V_i\| \leq \|I_H\| = 1$, and ψ_0 is normal since ψ_0 is the limit of a bounded increasing net in $Q_H(\mathcal{M})_n$, so that for any $h \in H$, $(\psi_0(\cdot)h, h) \in Q(\mathcal{M})_n$ by Ogasawara's theorem ([58; Th. 5]), i.e., $\psi_0 \in Q_H(\mathcal{M})_n$. ψ_0 is faithful since $F_H[\psi_0] = Q_H(\mathcal{M})_n$ by the maximality of $(\pi_i)_{i\in I}$ and Lemma 8.2. This completes the proof. ▮

Remark. Let $\psi_0 = \sum_{i\in I} \psi_i$ with $\psi_i = V_i^* \pi_i V_i$ be the faithful normal CP-state in the proof of Proposition 8.4, and assume that $\mathit{dim\ ess.\,supp}\,\psi_i = 1$ for each $i \in I$. Then, by Proposition 1.5.B, each ψ_i is regarded as a quasi-state $\psi_i = \omega_i(\cdot)P_i$, and ψ_0 is regarded as an "infinite sum" of $\omega_i \in Q(\mathcal{M})_n$. More precisely, consider $Tr(\psi_0(\cdot)) : \mathcal{M}^+ \to [0, +\infty]$ where Tr denotes the trace on the Hilbert space H, i.e., $Tr(X) = \sum_{\alpha\in\Lambda}(Xe_\alpha, e_\alpha)$ where $(e_\alpha)_{\alpha\in\Lambda}$ is an orthonormal basis of H. It is straightforward to show by the standard argument (e.g. [76]) that $Tr\,\psi_0$ is a faithful normal semi-finite weight. If $\mathcal{M}$ is σ-finite (i.e., $\alpha_\sigma(\mathcal{M}) = \aleph_0$) and $\sum_{i\in I} \|\omega_i\| = 1$, then $Tr\,\psi_0$ is a faithful normal state.

The above remark shows that weights appear here as the trace of infinite dimensional CP-states. We now generalize the GNS-construction to CP-states so that it includes the semi-cyclic representations associated with weights.

Let $\mathcal{A}$ be a C*-algebra and H be a Hillbert space. Assume that we are given a CP-state $\psi = V^*\pi V \in Q_H(\mathcal{A})$. We define

$$\mathcal{N} = \{a \in \mathcal{A};\ \psi(a^*a) = 0\}.$$

Then, $\mathcal{N}$ is a norm closed left ideal of $\mathcal{A}$, since, for $a \in \mathcal{A}$ and $b \in \mathcal{N}$,

$$\psi((ab)^*(ab)) = \psi(b^*a^*ab) \leq \|a\|^2\psi(b^*b) = 0,$$

so that $ab \in \mathcal{N}$. Consider the quotient space $\mathcal{A}/\mathcal{N}$, and define

$$(x,y)_\psi := Tr\,\psi(y^*x) \qquad \text{for } x,y \in \mathcal{A}/\mathcal{N}.$$

Similarly as the GNS-construction, we can show the following.

PROPOSITION 8.5. *In the situation described above,*

(1) $Tr\,\psi$ *is a weight, i.e., the mapping* $Tr\,\psi : \mathcal{A}^+ \to [0,+\infty]$ *satisfies*

(i) $Tr\,\psi(x+y) = Tr\,\psi(x) + Tr\,\psi(y)$

(ii) $Tr\,\psi(\lambda x) = \lambda Tr\,\psi(x) \quad \text{for } \lambda \geq 0$

(2) $(x,y)_\psi$ *defines an inner product on*

$$\begin{aligned}\mathcal{A}_\psi &:= \{x \in \mathcal{A}/\mathcal{N};\ Tr\,\psi(x^*x) < +\infty\}\\ &= \{x \in \mathcal{A}/\mathcal{N};\ \psi(x^*x) \in \mathcal{T}(H)\ (\textit{the trace class operators on } H)\}\end{aligned}$$

(3) $\mathcal{A}$ *is represented on the Hilbert space* $\mathcal{H}_\psi := \overline{\mathcal{A}_\psi}$ *(the completion of* $\mathcal{A}_\psi$ *by* $(\cdot,\cdot)_\psi$*) as the operators* $\pi_\psi(a)$ *(*$a \in \mathcal{A}$*) which are defined to be the continuous extension of the map*

$$a : x + \mathcal{N} \in \mathcal{A}_\psi \mapsto ax + \mathcal{N} \in \mathcal{A}_\psi.$$

PROOF: (1) and (2) are immediate. For (3), we note that, for $a \in \mathcal{A}$ and $x, y \in \mathcal{H}_\psi$,

$$\begin{aligned}(\pi_\psi(a^*a)x, y)_\psi &= Tr\,\psi(y^*\pi_\psi(a^*)\pi_\psi(a)x) \\ &\leq \|a\|^2 Tr\,\psi(y^*x) \\ &= \|a\|^2(x, y)_\psi,\end{aligned}$$

so that $\pi_\psi(a) \in B(\mathcal{H}_\psi)$ for $a \in \mathcal{A}^+$. By the linear combination $a = a_1 - a_2 + i(a_3 - a_4)$ with $a_i \in \mathcal{A}^+$ $(1 \leq i \leq 4)$, we conclude that $\pi_\psi(a) \in B(\mathcal{H}_\psi)$ for all $a \in \mathcal{A}$. ∎

DEFINITION 8.6. *We call the representation* $(\pi_\psi, \mathcal{H}_\psi)$ *constructed above the CP-GNS representation of* $\mathcal{A}$ *associated with the CP-state* $\psi \in Q_H(\mathcal{A})$.

Remarks. 1. If $\mathcal{A}$ is a W*-algebra $\mathcal{M}$, and $\psi \in Q_H(\mathcal{M})_n$ is a normal CP-state, then $Tr\,\psi$ is normal (i.e., $Tr\,\psi(\sup_i x_i) = \sup_i Tr\,\psi(x_i)$ for $x_i \in \mathcal{M}^+$), hence π_ψ is a normal representation.

2. π_ψ is not cyclic in general; but if $e \in \mathcal{H}_\psi$, then π_ψ is cyclic.

3. If $\psi \in Q(\mathcal{A})$ (a quasi-state, i.e., $dim\ ess.supp\,\psi = 1$), then the above procedure reduces to the usual GNS-construction.

4. ψ can be considered as a bounded linear operator from the Hilbert space $\mathcal{H}_\psi$ to the Hilbert space $\mathcal{HS}(H)$ (the class of Hilbert-Schmidt operators on H) with

the inner product $(X, Y) = Tr\, Y^*X$, since, for $x \in \mathcal{H}_\psi$,

$$\begin{aligned}\|\psi(x)\|^2_{\mathcal{HS}(H)} &= Tr\,(\psi(x)^*\psi(x)) = Tr\,(V^*\pi(x)V)^*(V^*\pi(x)V)\\ &\leq \|V\|^2 Tr\,(V^*\pi(x^*x)V) \leq \|V\|^2 Tr\,\psi(x^*x) = \|V\|^2\|x\|^2_\psi.\end{aligned}$$

If we denote the norm of this operator by $\S \cdot \S$, then $\S\psi\S \leq \|V\| = \|\psi\|^{\frac{1}{2}}$.

PROPOSITION 8.7. *Let $\mathcal{A}$ be a C^*-algebra and H be a Hilbert space. If $\psi \in Q_H(\mathcal{A})$ is a faithful CP-state, then $Tr\,\psi$ is a faithful weight.*

PROOF: Let $\psi = V^*\pi V$ be the canonical expression, and let $(e_i)_{i\in I}$ be an orthonormal basis of H. Observe that, for $a \in \mathcal{A}$,

$$\begin{aligned}\|V\|^2 Tr\,\psi(a^*a) &= \|V\|^2 \sum_{i\in I}(\psi(a^*a)e_i, e_i) = \|V\|^2 \sum_{i\in I}(V^*\pi(a^*)\pi(a)Ve_i, e_i)\\ &\geq \sum_{i\in I}((V^*\pi(a)V)^*(V^*\pi(a)V)e_i, e_i) = \sum_{i\in I}\|\psi(a)e_i\|^2.\end{aligned}$$

Assume that ψ is faithful and $Tr\,\psi(a) = 0$ for $a \in A^+$, then from the above inequality $\psi(a^{\frac{1}{2}})e_i = 0$ for all $i \in I$, i.e., $\psi(a^{\frac{1}{2}}) = 0$. Since ψ is faithful, $a^{\frac{1}{2}} = 0$, hence $a = 0$, so that $Tr\,\psi$ is a faithful weight. ∎

COROLLARY 8.8. *Any W^*-algebra has a faithful normal CP-GNS representation.*

PROOF: This follows from Proposition 8.4 and Proposition 8.7. ∎

Following [72], a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$ is said to be *standard* if there exists a conjugation $J : \mathcal{H} \to \mathcal{H}$ such that the mapping $x \mapsto Jx^*J$ is a *-anti-isomorphism from $\mathcal{M}$ to $\mathcal{M}'$ which acts identically on the center $Z(\mathcal{M}) =$

$\mathcal{M} \cap \mathcal{M}'$. The Tomita-Takesaki theory states that every W*-algebra has a standard representation; in view of the above arguments, this statement is equivalent to asserting the existence of a faithful normal CP-GNS representation.

Let $\mathcal{M} \subset B(\mathcal{H})$ be a standard von Neumann algebra. Then there exists a natural identification between $\mathcal{M}'$ and the opposite algebra $\mathcal{M}^\circ$, since the opposite algebra $\mathcal{M}^\circ$ is characterized by the existence of a *-anti-isomorphism from $\mathcal{M}$ to $\mathcal{M}^\circ$, and $\mathcal{M}'$ satisfies this condition by the map $x \in \mathcal{M} \mapsto Jx^*J \in \mathcal{M}'$. We shall use this fact to realize Tomita-Takesaki's anti-isomorphism theorem in our CP-geometrical setting. For this purpose, recalling the CP-duality theorem for $\mathcal{M}^\circ$(Theorem 7.3), we then need to express the standard structure in non-standard representations. The following Proposition solves this problem.

PROPOSITION 8.9. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a standard von Neumann algebra, and let π be a normal representation of $\mathcal{M}$ on H_π. Then, there exists a Hilbert space H_π^* and an anti-unitary operator $J_\pi : H_\pi \to H_\pi^*$ such that*

$$\pi^t(a) = J_\pi^* \pi(a)^* J_\pi \qquad \text{for all } a \in \mathcal{M}^\circ = \mathcal{M}',$$

where $\pi^t(a) = \pi(a)$ for all $\pi \in Rep_c(\mathcal{M})_n$ if and only if $a \in Z(\mathcal{M})$.

PROOF: We first consider the case that π is a normal cyclic representation with cyclic vector ξ. Let $\omega(\cdot) = (\pi(\cdot)\xi, \xi)$ be the state associated with π and ξ, and define

$$\mathcal{N}_l = \{a \in \mathcal{M};\ \omega(a^*a) = 0\}$$

$$\mathcal{N}_r = \{a \in \mathcal{M};\ \omega(aa^*) = 0\}$$

where we note that $\mathcal{N}_r = \mathcal{N}_l^*$. Consider the quotient spaces $\mathcal{M}/\mathcal{N}_l$ and $\mathcal{M}/\mathcal{N}_r$, and define

$$(x, y)_l := \omega(y^* x) \qquad \text{for } \ x, y \in \mathcal{M}/\mathcal{N}_l$$

$$(x, y)_r := \omega(x y^*) \qquad \text{for } \ x, y \in \mathcal{M}/\mathcal{N}_r$$

which define inner products on $\mathcal{M}/\mathcal{N}_l$ and $\mathcal{M}/\mathcal{N}_r$ respectively; we define the Hilbert spaces

$$\mathcal{H}_l = \overline{\mathcal{M}/\mathcal{N}_l} \qquad \text{and} \qquad \mathcal{H}_r = \overline{\mathcal{M}/\mathcal{N}_r}$$

as the completions by the inner products. Then, by the GNS-construction, $\pi(a)$ $(a \in \mathcal{M})$ is the continuous extension of the map

$$a : x + \mathcal{N}_l \in \mathcal{M}/\mathcal{N}_l \mapsto ax + \mathcal{N}_l \in \mathcal{M}/\mathcal{N}_l.$$

On the other hand, since $\mathcal{M}^\circ$ is regarded as $\mathcal{M}$ with the product being reversed, $\mathcal{M}^\circ$ is represented as the continuous extension of the map

$$a^\circ : x + \mathcal{N}_r \in \mathcal{M}/\mathcal{N}_r \mapsto xa + \mathcal{N}_r \in \mathcal{M}/\mathcal{N}_r$$

where $a \in \mathcal{M}$, which can be expressed as the left action

$$a^\circ : x + \mathcal{N}_r \in \mathcal{M}/\mathcal{N}_r \mapsto Ja^* Jx + \mathcal{N}_r \in \mathcal{M}/\mathcal{N}_r.$$

This map provides the representation

$$\pi : \mathcal{M}^\circ \to B(H_\pi)^\circ = B(\mathcal{H}_r)$$

in Definition 7.2.

We next define an anti-unitary operator $J_\pi : \mathcal{H}_l \to \mathcal{H}_r$ as the continuous extension of the map

$$x + \mathcal{N}_l \in \mathcal{M}/\mathcal{N}_l \mapsto Jx + \mathcal{N}_r \in \mathcal{M}/\mathcal{N}_r$$

Consider the following diagram;

$$\begin{tikzcd}
x + \mathcal{N}_l \in \mathcal{M}/\mathcal{N}_l \subset \mathcal{H}_l \arrow[r, "J_\pi"] \arrow[d, "\pi^t(a^\circ)"', "=\pi(a)"] & Jx + \mathcal{N}_r \in \mathcal{M}/\mathcal{N}_r \subset \mathcal{H}_r \arrow[d, "\pi(a^\circ)^*"] \\
ax + \mathcal{N}_l \in \mathcal{M}/\mathcal{N}_l \subset \mathcal{H}_l & (JaJ)Jx + \mathcal{N}_r \in \mathcal{M}/\mathcal{N}_r \subset \mathcal{H}_r \arrow[l, "J_\pi^*"']
\end{tikzcd}$$

where each arrow is defined on the dense sets, so that $\pi^t : \mathcal{M}^\circ = \mathcal{M}' \to B(\mathcal{H}_l)$ is given as the continuous extension of this cycle, i.e.,

$$\pi^t(a^\circ) = J_\pi^* \pi(a^\circ)^* J_\pi \qquad \text{for all } a^\circ \in \mathcal{M}^\circ = \mathcal{M}'.$$

Then our claim follows by setting $H_\pi = \mathcal{H}_l$ and $H_\pi^* = \mathcal{H}_r$.

If π is not a cyclic representation, then decompose π into cyclic representations, i.e., $\pi = \oplus_{i \in I} \pi_i$. Set $H_\pi = \oplus_{i \in I} H_{\pi_i}$, $H_\pi^* = \oplus_{i \in I} H_{\pi_i}^*$, and $J_\pi = \oplus_{i \in I} J_{\pi_i} : H_\pi \to H_\pi^*$.

To show the final statement, assume that $\pi^t(a^\circ) = \pi(a^\circ)$, i.e., $\pi(a) = \pi^t(a)$ for $a \in \mathcal{M}$ and all $\pi \in Rep_c(\mathcal{M})_n$. Then, by the CP-duality theorem (Proposition 7.3.B), $a \in \mathcal{M} \cap \mathcal{M}^\circ = \mathcal{M} \cap \mathcal{M}' = Z(\mathcal{M})$. ▮

Remark. From the above proof, it follows that a normal cyclic representation $\pi : \mathcal{M} \to B(H_\pi)$ is standard if and only if $\omega_\pi = (\pi(\cdot)\xi, \xi)$ is faithful, since it requires that J is a conjugation on $\mathcal{H}_l = \mathcal{H}_r$ which is equivalent to asserting $\mathcal{N}_l = \mathcal{N}_r = 0$.

The following theorem gives the CP-geometric expression of Tomita-Takesaki's anti-isomorphism theorem ([74]).

THEOREM 8.10. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a standard von Neumann algebra and let H be a Hilbert space with $dim H \geq \alpha_c(\mathcal{M})_n$. Then*

$$\mathcal{M} \cong AB(Q_H(\mathcal{M})_n, B(H)) \cong A^t B(Q_H^t(\mathcal{M})_n, B(H))$$

$$\mathcal{M}' \cong A^t B(Q_H(\mathcal{M})_n, B(H)) \cong AB(Q_H^t(\mathcal{M})_n, B(H))$$

and

$$Z(\mathcal{M}) \cong AB(Q_H(\mathcal{M})_n, B(H)) \cap A^t B(Q_H(\mathcal{M})_n, B(H))$$

where the transpose of $\psi = V^*\pi V \in Q_H(\mathcal{M})_n$ is defined by $\psi^t = V^* J_\pi^* \pi^* J_\pi V$.

PROOF: From Proposition 8.9, $\pi(a^\circ) = \pi^t(a) = J_\pi^* \pi(a)^* J_\pi$ for $a^\circ \in \mathcal{M}^\circ = \mathcal{M}'$, so that

$$a^\circ \in \mathcal{M}' : \psi = V^*\pi V \in Q_H(\mathcal{M})_n \mapsto V^* J_\pi^* \pi(a^*) J_\pi V = \psi^t(a)$$

Then, the theorem follows from Proposition 7.3.B and Proposition 8.9. ∎

Remark. In the above theorem, for $\gamma \in AB(Q_H(M)_n, B(H))$ which corresponds to $a \in \mathcal{M}$, and $\psi = V^*\pi V \in Q_H(\mathcal{M})_n$, set

$$\gamma : \psi \in Q_H(\mathcal{M})_n \mapsto \psi(a) = V^*\pi(a)V \in B(H)$$

$$J\gamma J : \psi \in Q_H(\mathcal{M})_n \mapsto \psi^t(a^*) = V^* J_\pi^* \pi(a) J_\pi V \in B(H),$$

then the standard relations $\mathcal{M}' = J\mathcal{M}J$ and $\mathcal{M} = J\mathcal{M}'J$ follows immediately. Note also that the standard *-anti-isomorphism $x \in \mathcal{M} \mapsto Jx^*J \in \mathcal{M}'$ induces the transposed CP-affine isomorphism

$$\psi = V^*\pi V \in Q_H(\mathcal{M})_n \mapsto \psi^t = V^* J_\pi^* \pi^* J_\pi V \in Q_H(\mathcal{M}')_n$$

as the CP-dual map.

Based on this CP-geometric expression of Tomita-Takesaki theory, we proceed to express Connes' orientability condition in our CP-geometric context. In the following, $\mathcal{H}$ denotes a complex Hilbert space (which should be distinguished from the Hilbert space H of CP-duality theorems), and $\mathcal{H}^+$ denotes a self-dual cone in $\mathcal{H}$, i.e., $\mathcal{H}^+$ is characterized by

$$\mathcal{H}^+ = \{\xi \in \mathcal{H};\ (\xi, \eta) \geq 0 \ \text{ for all } \ \eta \in \mathcal{H}^+\}.$$

If $\mathcal{M} \subset B(\mathcal{H})$ is a standard σ-finite von Neumann algebra with cyclic and separating vector ξ and modular operator Δ, then $\mathcal{P} = \overline{\Delta^{\frac{1}{4}} \mathcal{M}^+ \xi}^{\|\cdot\|}$ is an example of a self-dual cone, called the natural self-dual cone, which was studied in [10], [24] and [41]. Connes [24] characterized the natural cone $\mathcal{P}$ (in the sense of Haagerup [41]) by the three properties: self-duality, facial homogeneity and orientability. The definitions of the last two concepts are recalled here for convenience.

Let $D(\mathcal{H}^+)$ denote the set of all *derivations* of $\mathcal{H}^+$, i.e.,

$$D(\mathcal{H}^+) = \{\delta \in B(\mathcal{H});\ e^{t\delta} \in B(\mathcal{H}^+) \ \text{ for all } \ t \in \mathbf{R}\}$$

which is self-adjoint and has a structure of real Lie algebra (cf. [24]). A self-dual cone $\mathcal{H}^+$ is defined to be *facially homogeneous* if, for any face F of $\mathcal{H}^+$, we have $N_F := P_F - P_{F^\perp} \in D(\mathcal{H}^+)$ where P_F [resp. $P_{F^\perp}$] is the orthogonal projection onto the closed subspace spanned by F [resp. $F^{\perp} := \{\xi \in \mathcal{H}^{+};\ (\xi, \eta) = 0$ for any $\eta \in F\}$] (cf. [24]). In the series of work [13-16], [46], Bellissard, Iochum and Lima showed that a self-dual cone $\mathcal{H}^{+}$ is facially homogeneous if and only if $M = D(\mathcal{H}^{+})_{sa}$ has the structure of a JBW-algebra (see, in particular, [46; Ch.VII]).

In this respect, another question of orientability arises. Connes [24] gave the following orientability condition for $D(\mathcal{H}^{+})$.

DEFINITION 8.11 ([24]). *A self-dual cone $\mathcal{H}^{+}$ is said to be **orientable** , if $\underline{D}(\mathcal{H}^{+}) := D(\mathcal{H}^{+})/ZD(\mathcal{H}^{+})$ (where $ZD(\mathcal{H}^{+})$ denotes the center of $D(\mathcal{H}^{+})$ as a Lie algebra) has a structure of a complex Lie algebra, i.e., there exists a map $I : \underline{D}(\mathcal{H}^{+}) \to \underline{D}(\mathcal{H}^{+})$ (which we call **Connes' orientation map**) such that*

i) $I^{2} = -1$

ii) $[I\underline{\delta}_1, \underline{\delta}_2] = [\underline{\delta}_1, I\underline{\delta}_2] = I[\underline{\delta}_1, \underline{\delta}_2]$ *for any* $\delta_i \in D(\mathcal{H}^{+})$ $(i = 1, 2)$

iii) $I(\underline{\delta}^{*}) = -I(\underline{\delta})^{*}$

Combined with Bellissard-Iochum's context, Connes's results shows that a facially homogeneous self-dual cone $\mathcal{H}^{+}$ is orientable if and only if the JBW-algebra $D(\mathcal{H}^{+})_{sa}$ is Jordan isomorphic to the self-adjoint part of a W*-algebra.

In what follows, let us restrict ourselves to the case where $M = D(\mathcal{H}^{+})_{sa}$ is a JW-algebra, since we are now interested in the orientability condition. Then, we shall interpret the Connes' orientability map in our CP-geometrical setting.

Let M be a universally reversible JW-algebra and H be a Hilbert space with $dim H \geq \alpha_c(M)_n$. Then, by [46], there exists a facially homogeneous self-dual cone

$\mathcal{H}^+$ such that $M = D(\mathcal{H}^+)_{sa}$. By the CP-duality theorem (Proposition 7.9.B),

$$M \cong AB(\tilde{Q}_H(M)_n, B(H)_{sa}) \qquad \text{(Jordan isomorphism)}$$

Suppose that $\tilde{Q}_H(M)_n$ is CP-orientable, i.e., there exists a W*-algebra $\mathcal{M}$ such that $M \cong \mathcal{M}_{sa}$. Then, by Theorem 7.10.B, any $\psi \in Q_H^s(W^*(M))_n$ is uniquely expressed as $\psi = \frac{1}{2}(\varphi + \varphi^t)$ with $\varphi \in Q_H(\mathcal{M})_n$. We use the following functional representation of the enveloping W*-algebra $W^*(M)$ of M on $Q_H^s(W^*(M))_n$.

Proposition 8.12. *Let $M = \mathcal{M}_{sa}$ for some W*-algebra $\mathcal{M}$ and let H be a Hilbert space with $dim H \geq \alpha_c(M)_*$. Then,*

$$W^*(M) \cong \{\gamma_{a\oplus b^\circ} : \frac{1}{2}(\varphi + \varphi^t) \in Q_H^s(W^*(M))_n \mapsto \frac{1}{2}(\varphi(a) + \varphi^t(b)); \quad a, b \in \mathcal{M},\ a - b \in \mathcal{M}_{nab}\}$$

$$M \cong \{\gamma_{a\oplus a^\circ} : \frac{1}{2}(\varphi + \varphi^t) \in Q_H^s(W^*(M))_n \mapsto \frac{1}{2}(\varphi(a) + \varphi^t(a)),\ a \in \mathcal{M}_{sa}\}$$

Proof: This follows as the functional expression of

$$W^*(M) = \{a \oplus b^\circ \in \mathcal{M} \oplus \mathcal{M}^\circ;\ a - b \in \mathcal{M}_{nab}\}$$

where M is embedded as

$$M = \{a \oplus a^\circ \in \mathcal{M} \oplus \mathcal{M}\}$$

(cf.[45; 7.4.15]). ∎

In the above Proposition, we can assume that the W*-algebra $\mathcal{M}$ is represented as a standard von Neumann algebra, then we can set $\pi^t = J_\pi^* \pi^* J_\pi$ by Proposition 8.9.

Our next step is to define a complexification of this functional representation of M, and in this respect we naturally realize that there are two types of complexifications, i.e.,

$$I_{+}\gamma_a : \psi = \frac{1}{2}(\varphi + \varphi^t) \mapsto \frac{1}{2}(i\varphi(a) + i\varphi^t(a))$$

and

$$I_{-}\gamma_a : \psi = \frac{1}{2}(\varphi + \varphi^t) \mapsto \frac{1}{2}(i\varphi(a) - i\varphi^t(a))$$

It is obvious that I_+ corresponds to the complexification of M into the W*-algebra $\mathcal{M}$, i.e., $I_+ = i$. I_- can be understood as the non-flip complexification of I_+, which is equivalent to setting $\varphi^t = V^* J_\pi^* \pi J_\pi V$, replacing π^* by π in the definition of $\psi^t = V^* J_\pi^* \pi^* J_\pi V$, or defining $\varphi^t(a) = V^* J_\pi^* \pi^t(a^*) J_\pi V$ for all $a \in \mathcal{M}$. Consequently, we conclude that I_- corresponds to the complexification to yield $D(\mathcal{H}^+)$, since if $D(\mathcal{H}^+) = M + I_+ M = M + iM = \mathcal{M}$, then it gains a complex Lie structure; note that this is not true in the general case. Hence, we have the functional representation of $\delta \in D(\mathcal{H}^+)$ on $Q_H^s(W^*(M))_n$ by

$$\delta = \delta_x : \frac{1}{2}(\varphi + \varphi^t) \mapsto \frac{1}{2}(\varphi(x) + \varphi^t(x^*)) \quad \text{with } x \in \mathcal{M}$$

PROPOSITION 8.13. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a standard von Neumann algebra. Then,*

$$D(\mathcal{H}^+) = \{\delta_x = x + JxJ;\ x \in \mathcal{M}\}$$

PROOF: From the CP-duality of $\mathcal{M}$ and $\mathcal{M}'$ (Theorem 8.10) and the above functional representation, we have $\delta_x = x + JxJ$ with $x \in \mathcal{M}$. ∎

Note that the proof of the above expression of $D(\mathcal{H}^+)$ naturally came out from the setting of CP-duality, which simplifies the original arguments in [24] or [46]. In this representation, I_- is expressed as

$$I_- : \delta_x = x + JxJ \mapsto \delta_{ix} = ix + JixJ = ix - iJxJ.$$

It is immediate to see that $kerI_- = Z(M)$. In fact, let $\delta_x \in kerI_-$ with $x \in \mathcal{M}$, then

$$I_-\delta_x = ix + JixJ = i(x - JxJ) = 0,$$

i.e., $x = JxJ$, which implies $x \in Z(\mathcal{M}_{sa})$, hence $\delta_x \in Z(M)$. From this, it follows that I_- is well-defined on $\underline{D}(\mathcal{H}^+) = D(\mathcal{H}^+)/ZD(\mathcal{H}^+)$.

PROPOSITION 8.14. *$\underline{D}(\mathcal{H}^+) = D(\mathcal{H}^+)/ZD(\mathcal{H}^+)$ is a complex Lie algebra with the complexification map $I = I_-$.*

PROOF: We have to show the three properties in Definition 8.11.

i) $I^2 = -1$ is immediate.

ii) We will show $[I(\underline{\delta}_1), \underline{\delta}_2] = I[\underline{\delta}_1, \underline{\delta}_2]$ for $\delta_1, \delta_2 \in D(\mathcal{H}^+)$. By Proposition 8.13, there exists $x_1, x_2 \in \mathcal{M}$, such that $\underline{\delta}_i \ni \delta_{x_i} = x_i + Jx_iJ$ $(i = 1, 2)$. Noting $Jx_iJ \in \mathcal{M}'$, we have

$$\begin{aligned}[I\delta_{x_1}, \delta_{x_2}] &= [ix_1 - iJx_1J, x_2 + Jx_2J] \\ &= [ix_1, x_2] - [iJx_1J, Jx_2J] \\ &= i[x_1, x_2] - iJ[x_1, x_2]J \\ &= I[\delta_{x_1}, \delta_{x_2}].\end{aligned}$$

Similarly, we can prove $[\underline{\delta}_1, I(\underline{\delta}_2)] = I[\underline{\delta}_1, \underline{\delta}_2]$.

iii) To prove $I(\delta^*) = -I(\delta)^*$ for $\delta \in D(\mathcal{H}^+)$, let $\delta = \delta_x = x + JxJ$ with $x \in \mathcal{M}$, then we have

$$I(\delta_x^*) = ix^* - iJx^*J = -(ix - iJxJ)^* = -I(\delta_x)^*.$$

∎

Thus we realized Connes's orientation map in our CP-geometrical setting. In the above functional representation, the correspondence between $\mathcal{M}$ and $D(\mathcal{H}^+)$ is given as follows.

PROPOSITION 8.15.

$$\mathcal{M} = (1 - iI)D(\mathcal{H}^+)$$

$$\mathcal{M}' = (1 + iI)D(\mathcal{H}^+)$$

PROOF: By Proposition 8.13, let $\delta = x + JxJ \in D(\mathcal{H}^+)$ with $x \in \mathcal{M}$. Then,

$$(1 - iI)\delta = x + JxJ - i(ix - iJxJ) = x \in \mathcal{M},$$

and

$$(1 + iI)\delta = x + JxJ + i(ix - iJxJ) = JxJ \in \mathcal{M}'.$$

Tomita-Takesaki's anti-isomorphism theorem completes the proof. ∎

COROLLARY 8.16. *The von Neumann algebra generated by* $D(\mathcal{H}^+)$ *is* $(\mathcal{M} \cup \mathcal{M}')''$, *which is isomorphic to* $W^*(M)$.

PROOF: From Prop. 8.15, the von Neumann algebra generated by $D(\mathcal{H}^+)$ should be generated by $\mathcal{M}$ and $\mathcal{M}' \cong M^\circ$ (since $\mathcal{M}$ is a standard von Neumann algebra), which, by definition, is isomorphic to $W^*(M)$, since $M = \mathcal{M}_{sa}$ is a JBW-algebra of complex type (cf. [4; Prop.5.11]). ∎

Connes [24] showed that $\mathcal{H}^+$ is recovered as the natural self-dual cone of the von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$, i.e., $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ is a hyperstandard von Neumann algebra in the sense of [41], i.e.,

$$i)\ \ J\mathcal{M}J = \mathcal{M}'$$

$$ii)\ \ JxJ = x^* \quad \text{for all } x \in Z(\mathcal{M})$$

$$iii)\ \ J\xi = \xi \quad \text{for all } \xi \in \mathcal{H}^+$$

$$iv)\ \ x(JxJ)\mathcal{H}^+ \subset \mathcal{H}^+ \quad \text{for all } x \in \mathcal{M}$$

i) and *ii*) are obvious from our arguments. See [24; Th 5.2] or [46;IV.1.2] for *iii*) and *iv*).

Now let us study the structure of $D(\mathcal{H}^+)$ in terms of derivations.

PROPOSITION 8.17. *Let $\mathcal{M}$ be a standard von Neumann algebra, and let $\mathcal{D}(\mathcal{M})$ denote the set of all bounded derivations of $\mathcal{M}$, i.e.,*

$$\mathcal{D}(\mathcal{M}) = \{d \in B(\mathcal{M});\ d(xy) = x(dy) + (dy)x, \quad x, y \in \mathcal{M}\}$$

Then, there exists one to one correspondences between $\mathcal{D}(\mathcal{M})$, $\mathcal{M}$ and $\mathcal{D}(\mathcal{H}^+)$ given by

$$\mathrm{ad}(x) \in \mathcal{D}(\mathcal{M}) \leftrightarrow x \in \mathcal{M} \leftrightarrow \delta_x = x + JxJ \in D(\mathcal{H}^+)$$

where $\mathrm{ad}(x)(a) = [x, a]$ for $a \in \mathcal{M}$.

PROOF: It is fundamental that every $d \in \mathcal{D}(\mathcal{M})$ is inner, i.e., there exists $x \in \mathcal{M}$ such that $d = \mathrm{ad}(x)$. (cf. [65; Th.4.1.6]). The rest is already proved in Proposition 8.13. ∎

Remark. We can directly see the correspondence $\mathrm{ad}(x) \in \mathcal{D}(\mathcal{M}) \leftrightarrow \delta_x \in D(\mathcal{H}^+)$ as follows. Set $U(x;t) = e^{tx}Je^{tx}J$, then $U(x;t)$ is a one-parameter semigroup such that $U(x;t)\mathcal{H}^+ \subset \mathcal{H}^+$, whose generator is

$$\begin{aligned}\delta_x &= \lim_{t\to 0}(e^{tx}Je^{tx}J - 1)/t \\ &= \lim_{t\to 0}\{((e^{tx}-1)/t)Je^{tx}J + J(e^{tx}-1)/tJ\} \\ &= x + JxJ.\end{aligned}$$

Hence, $U(x;t) = e^{t\delta_x} \in B(\mathcal{H}^+)$ for all $t \in \mathbf{R}$, i.e., $\delta_x = x + JxJ \in D(\mathcal{H}^+)$.

Let $\mathcal{M}$ be a W*-algebra. There exists a one to one correspondence between the set $*\mathcal{D}(\mathcal{M})$ of all bounded *-derivations of a W*-algebra $\mathcal{M}$ (i.e., satisfying $\delta(a^*) = \delta(a)^*$ for all $a \in \mathcal{M}$), and the set $Aut(\mathcal{M})$ of all *-automorphisms on $\mathcal{M}$

$$\mathrm{ad}(ih) \in *\mathcal{D}(\mathcal{M}) \leftrightarrow \alpha_t(h) = e^{-ith}(\cdot)e^{ith} \in Aut(\mathcal{M}),$$

where $h \in \mathcal{M}_{sa}$. Then, each *-automorphism α_t induces a CP-affine automorphism $\sigma_t(h)$ as the CP-dual on the normal CP-state space $Q_H(\mathcal{M})_n$ by

$$\sigma_t(h) = \alpha_t(h)^\natural : V^*\pi V \in Q_H(\mathcal{M})_n \mapsto V^*e^{-it\pi(h)}\pi e^{it\pi(h)}V \in Q_H(\mathcal{M})_n.$$

Applying these arguments to a standard von Neumann algebra $\mathcal{M}$ and $D(\mathcal{H}^+)$, we have the following CP-description of derivations and automorphism groups. We denote by $D(\mathcal{M})$ the set of all generators of $*\mathcal{D}(\mathcal{M})$.

PROPOSITION 8.18. *(1) Let $\mathcal{M}$ be a W^*-algebra and H be a Hilbert space with $dim H \geq \alpha_c(\mathcal{M})_*$. Then,*

(i) $D(\mathcal{M}) \cong \mathcal{M}_{sa} \cong AB(Q_H(\mathcal{M})_n, B(H)_{sa})$

(ii) $CP - Aut(\mathcal{M})_n = Aut(\mathcal{M})^\natural = \{\sigma_t(h); h \in D(\mathcal{M})\}$

(2) If $\mathcal{M}$ is a standard von Neumann algebra, then

(i) $D(\mathcal{H}^+)_{sa} \cong \mathcal{M}_{sa} \cong AB(\tilde{Q}_H(\mathcal{M}_{sa})_n, B(H)_{sa})$

(ii) $CP - Aut(\tilde{Q}_H(\mathcal{M}_{sa})) = \{\sigma_t(\delta) = \frac{1}{2}(\sigma_t(h) + \sigma_{-t}(h))$ *where* $\delta = h + JhJ$ *with* $h \in \mathcal{M}_{sa}$ $\}$.

PROOF: (1) and (2) (i) are immediate from CP-duality and our arguments.

To prove (2) (ii), let $\psi = \frac{1}{2}(\varphi + \varphi^t) \in Q_H^s(W^*(M))_n$ with $\varphi \in Q_H(\mathcal{M})_n$. Noting that $i\delta = ih + J(ih)^*J$, we have

$$\begin{aligned}
\sigma_t(\delta)(\psi)(a) &= \psi(e^{-it\delta} a e^{it\delta}) \\
&= \frac{1}{2}(V^*\pi(e^{-ith} a e^{ith})V + V^*\pi(e^{(ith)^\circ} a^\circ e^{(-ith)^\circ})V) \\
&= \frac{1}{2}(V^* e^{-it\pi(h)} \pi(a) e^{it\pi(h)} + V^* e^{it\pi^t(h)} \pi^t(a) e^{-it\pi^t(h)} V) \\
&\simeq \frac{1}{2}(\sigma_t(h) + \sigma_{-t}(h))(\psi(a))
\end{aligned}$$

∎

Remark. These arguments, which we discussed for W*-algebras so far, can be generalized for JBW-algebras to some extent. U. Haagerup and H. Hanche-Olsen [42] generalized Tomita-Takesaki's automorphism group for σ-finite JBW-algebras, and using this one parameter group, they generalized the correspondence $x \in M \mapsto \delta_x \in D(\mathcal{H}^+)_{sa}$ (cf. Proposition 8.13).

It seems that, for the existence of a modular group, the restriction to σ-finite case is not essential for universally reversible JW-algebras. In fact, let M be a universally reversible JW-algebra. Then, we can identify M as the image $\phi_M(M)$ of the canonical injection ϕ_M of M into the enveloping W*-algebra $W^*(M)$ such that $M = W^*(M)^\Phi$. Let ω_0 be a faithful normal semi-finite weight on $W^*(M)$, and $\sigma_t = \Delta^{-it}(\cdot)\Delta^{it}$ be the modular group with respect to ω_0, where we can assume that ω_0 is Φ-invariant; if not, we can redefine $\tilde{\omega}_0 = \frac{1}{2}(\omega_0 + \omega_0 \circ \Phi)$. By an argument similar to that in [42], one can check that ω_0 satisfies the KMS-condition with respect to the one parameter group $\Phi \circ \sigma_{-t} \circ \Phi \quad (t \in \mathbb{R})$, so that $\sigma_t = \Phi \circ \sigma_{-t} \circ \Phi$. We can then define

$$\tau_t = \frac{1}{2}(\sigma_t + \sigma_{-t})|_M,$$

which is shown easily to leave M invariant.

CHAPTER 9

STONE-WEIERSTRASS THEOREM FOR SEPARABLE C*-ALGEBRAS

We are going to apply our theory of CP-convexity to the general Stone-Weierstrass problem, i.e., to realize the non-commutative generalization of the classical Stone-Weierstrass theorem, which is translated in the non-commutative case as follows: Let A be a unital C*-algebra and let B be a C*-subalgebra of A which contains the identity of A, and suppose that B separates the pure states $P(A)$ of A, then $A = B$.

There have been many contributions to this problem, and it is known that the conjecture is true under some additional conditions (see, e.g., [1], [2], [9], [21], [22], [40], [54], [57], [62], [64], [69]). In this chapter, we will give another consideration to this problem from the point of view of CP-convexity and prove the general Stone-Weierstrass theorem for separable C*-algebras, i.e.,

THEOREM 9.1. *Let A be a unital and separable C*-algebra and let B be a C*-subalgebra of A which contains the identity of A, and suppose that B separates the pure states $P(A)$ of A. Then* $A = B$.

In our previous papers ([35], [38]), we generalized the setting of the problem to the cone of completely positive maps $CP(A, B(H))$ and examined several equivalent conditions of the conjecture, of which we shall use the following equivalent conditions throughout this chapter.

THEOREM 9.2 ([35], [38], [22]). *Let A be a unital C^*-algebra and let B be a C^*-subalgebra of A which contains the identity of A. Then the following conditions are equivalent.*

(*i*) *B separates $P(A)$.*

(*ii*) *B separates $P_H(A)$ for any Hilbert space H.*

(*iii*) *B separates $Irr(A)$.*

We shall use the following results which were proved by J. Anderson and J.W. Bunce [9].

THEOREM 9.3 ([9; COROLLARY 7]). *Let A be a unital and separable C^*-algebra, and let B be a C^*-subalgebra of A which contains the identity of A, and suppose that B separates the pure states $P(A)$ of A. Let $\omega \in S(B)$ be a state of B. $\pi_\omega : B \to B(H_{\pi_\omega})$ be the GNS-representation corresponding to ω, and let $\pi : A \to B(H_{\pi_\omega})$ extend π_ω, i.e., $\pi|_B = \pi_\omega$. Then, π is the unique extension of π_ω if and only if $\pi(A)'' = \pi_\omega(B)''$.*

THEOREM 9.4 ([9; THEOREM 8]). *Under the same conditions as above, if, for each $\omega \in S(B)$, there exists a unique representation $\tilde{\pi}_\omega : A \to B(H_{\pi_\omega})$ that extends π_ω (i.e., $\tilde{\pi}|_B = \pi_\omega$), then $A = B$.*

From these theorems, the proof of Theorem 9.1 is reduced to the following assertion, which we are going to prove in the remainder of this chapter.

PROPOSITION 9.5. *Let A and B satisfy the conditions of Theorem 9.1. Then, for each $\omega \in S(B)$, there exists a representation $\tilde{\pi}_\omega : A \to B(H_{\pi_\omega})$ which extends π_ω (i.e., $\tilde{\pi}_\omega|_B = \pi_\omega$) such that $\tilde{\pi}_\omega(A)'' = \pi_\omega(B)''$.*

PROOF: Let $\omega \in S(B)$ be arbitrary. We choose a maximal abelian subalgebra $\mathcal{M}_{\pi_\omega}$ of $\pi_\omega(B)'$ and keep it fixed in the subsequent arguments.

Since B is separable, H_{π_ω} is separable, so that there exists a countably Hilbert nuclear space Ω such that $\Omega \subset H_{\pi_\omega} \subset \Omega'$ is a rigged Hilbert space. Let H be a separable Hilbert space and let $\psi = V^*\pi_\omega V \in CP(B, B(H))$ be a pre-nuclear CP-map, i.e., $V \in B(H, \Omega_r)$ (see Chapter 6 for the definition of Ω_r). From the proof of the CP-decomposition theorem (Theorem 6.3), ψ is decomposed as

$$\psi = \int_Z V(\zeta)^* \pi_\omega(\zeta) V(\zeta)\, d\mu(\zeta) \qquad \text{(BW-integral)}$$

where (Z, μ) is the standard measure space corresponding to $\mathcal{M}_{\pi_\omega}$, $\pi_\omega(\zeta) \in Irr(B)$ and $V(\zeta) = P(\zeta)V \in B(H, H_{\pi_\omega(\zeta)})$. (Recall here that $P(\zeta) : \Omega_r \to H_{\pi_\omega}(\zeta)$ is a nuclear operator. See Chapter 6 for the definition of $P(\zeta)$.)

Note that, by Theorem 9.2, B separates $Irr(A)$, from which it follows that the restriction map $r : \pi \in Irr(A) \mapsto \pi|_B \in Irr(B)$ is bijective. Hence it would be reasonable to try to define an extension $\tilde{\psi}$ of ψ to A by

$$\tilde{\psi} = \int_Z V(\zeta)^* \tilde{\pi}_\omega(\zeta) V(\zeta)\, d\mu(\zeta)$$

where $\tilde{\pi}_\omega(\zeta)$ is the unique extension of $\pi_\omega(\zeta)$ to A (i.e., $\tilde{\pi}_\omega(\zeta) = r^{-1}(\pi_\omega(\zeta))$), if the integral is meaningful.

To ensure that this extension $\tilde{\psi}$ is indeed well-defined, we have to prove the measurability of the integrand. The integrability will then follow as we proved in Chapter 6.

LEMMA 9.6. *Under the condition above,* $\psi(\zeta) := V(\zeta)^*\tilde{\pi}_\omega(\zeta)V(\zeta)$ *is BW-measurable, i.e., the function*

$$\Psi(\zeta)(a;\xi,\eta)(\zeta) := (V(\zeta)^*\tilde{\pi}_\omega(\zeta)(a)V(\zeta)\xi,\eta)$$

is μ*-measurable for all* $a \in A$, *and* $\xi,\eta \in H$.

PROOF: We define, for $b \in B$, $f,g \in \Omega_r$ and $\zeta \in Z$,

$$F(b;f,g)(\zeta) := (P(\zeta)^*\pi_\omega(\zeta)(b)P(\zeta)f,g),$$

and consider the following diagram;

$$\begin{tikzcd}
(A) & \zeta \in Z \arrow[r, mapsto, "P^*\tilde{\pi}_\omega P"] & P(\zeta)^*\tilde{\pi}_\omega(\zeta)P(\zeta) \in P_{\Omega_r}(A) \arrow[r, mapsto, "(a;V\xi,V\eta)"] \arrow[d, leftrightarrow, "r"', "r^{-1}"] & \Psi(a;\xi,\eta)(\zeta) \in \mathbb{C} \\
(B) & \zeta \in Z \arrow[r, mapsto, "P^*\pi_\omega P"] & P(\zeta)^*\pi_\omega(\zeta)P(\zeta) \in P_{\Omega_r}(B) \arrow[r, mapsto, "(b;f,g)"] & F(b;f,g)(\zeta) \in \mathbb{C}
\end{tikzcd}$$

where $a \in A$, $b \in B$, $\xi,\eta \in H$ and $f,g \in \Omega_r$, and $(a;V\xi,V\eta)$ and $(b;f,g)$ are defined by

$$(a;V\xi,V\eta)(\varphi) := (\varphi(a)V\xi,V\eta) \qquad for\ \varphi \in CP(A,B(\Omega_r))$$

and

$$(b;f,g)(\varphi) := (\varphi(b)f,g) \qquad for\ \varphi \in CP(B,B(\Omega_r))$$

respectively. Note here that, B separates $P_{\Omega_r}(A)$ (cf. Theorem 9.2), so that there exists one to one correspondence $P_{\Omega_r}(A) \underset{r^{-1}}{\overset{r}{\leftrightarrows}} P_{\Omega_r}(B)$ by the restriction map r.

Then, from the above diagram,

$$\Psi(a;\xi,\eta)(\zeta) = (a; V\xi, V\eta) \circ r^{-1} \circ P^*\pi_\omega P.$$

Thus, the measurability of $\Psi(a;\xi,\eta)(\zeta)$ will be proved, when we show the following:

(1) $P^*\pi_\omega P : Z \to P_{\Omega_r}(B)$ is $\mathcal{B}(Z) - \mathcal{B}_{BW}(P_{\Omega_r}(B))$ measurable.

(2) $r^{-1} : P_{\Omega_r}(B) \to P_{\Omega_r}(A)$ is BW-continuous.

(3) $(a; V\xi, V\eta)$ is BW-continuous.

In the above, $\mathcal{B}(Z)$ is the standard Borel space on (Z,μ), and $\mathcal{B}_{BW}(P_{\Omega_r}(B))$ denotes the Borel space generated by the relative topology in $P_{\Omega_r}(B)$ induced from the BW-topology of $CP(B, B(\Omega_r))$. Since (3) is obvious from the definition of the BW-topology, we will show (1) and (2).

(1) : We first note that the row (B) of the diagram

$$F(b; f, g) = (b; f, g) \circ (P^*\pi_\omega P) \qquad (b \in B, f, g \in \Omega_r)$$

is μ-measurable, since $F(b;f,g)(\zeta) = (P(\zeta)^*\pi_\omega(\zeta)P(\zeta)f, g)$ is a μ-measurable function of $\zeta \in Z$ (cf. Proof of Theorem 6.3).

Recall that the BW-topology of $CP(B, B(\Omega_r))$ is generated by the sub-base

$$u_\delta(\phi; b; f, g; \epsilon) = \{\varphi \in CP(B, B(\Omega_r)) \cap S_\delta \, ; \, |((\varphi - \phi)(b)f, g)| < \epsilon\}$$

where $\phi \in CP(B, B(\Omega_r))$, $b \in B$, $f, g \in \Omega_r$ and $S_\delta = \{\varphi \in CP(B, B(\Omega_r)); \|\varphi\| \leq \delta\}$ for $\delta > 0$ (cf. [11], [59]), i.e., for any open set $\mathcal{O}$ in $CP(B, B(\Omega_r))$,

$$\mathcal{O} \cap S_\delta = \bigcup_{\alpha \in \Lambda_\delta} (\bigcap_{i=1}^{m_\alpha} u_\delta(\phi_i^{(\alpha)}; b_i^{(\alpha)}; f_i^{(\alpha)}, g_i^{(\alpha)}; \epsilon_i^{(\alpha)}))$$

where m_α is a positive integer for all $\alpha \in \Lambda_\delta$,and Λ_δ can be an uncountable set in general. From our assumption that B is a separable C*-algebra, and Ω_r is a separable Hilbert space, it follows that $CP(B, B(\Omega_r)) \cap S_\delta$ with BW-topology is a Polish space for all $\delta > 0$ (by the same argument of [28;Prop. 3.7.1.]), hence it has the Lindelöf property, so that Λ_δ is actually a countable set, i.e.,

$$\mathcal{O} \cap S_\delta = \bigcup_{k=1}^{\infty} (\bigcap_{i=1}^{m_k} u_\delta(\phi_i^{(k)}; b_i^{(k)}; f_i^{(k)}, g_i^{(k)}; \epsilon_i^{(k)}))$$

.

Let us take an arbitrary relatively open set G in $P_{\Omega_r}(B)$. Then, there exists a BW-open set $\mathcal{O}_G$ in $CP(B, B(\Omega_r))$ such that, for all $\delta > 0$,

$$\begin{aligned} G \cap S_\delta &= (\mathcal{O}_G \cap P_{\Omega_r}(B)) \cap S_\delta = (\mathcal{O}_G \cap S_\delta) \cap P_{\Omega_r}(B) \\ &= \bigcup_{k=1}^{\infty} (\bigcap_{i=1}^{m_k} u_\delta(\phi_i^{(k)}; b_i^{(k)}; f_i^{(k)}, g_i^{(k)}; \epsilon_i^{(k)}) \cap P_{\Omega_r}(B)). \end{aligned}$$

We will show that

$$(P^* \pi_\omega P)^{-1}(u_\delta(\phi; b; f, g; \epsilon) \cap P_{\Omega_r}(B)) \in \mathcal{B}(Z)$$

for all $\delta > 0$, $\phi \in CP(B, B(\Omega_r))$, $b \in B$, $f, g \in \Omega_r$ and $\epsilon > 0$, which then proves that $P\pi_\omega P$ is $\mathcal{B}(Z)$–$\mathcal{B}_{BW}(P_{\Omega_r}(B))$ measurable, since the sets of the form $u_\delta(\phi; b; f, g; \epsilon) \cap P_{\Omega_r}(B)$ countably generates the topology of $P_{\Omega_r}(B)$ as we have shown above.

We first consider the special case $\phi = 0$. Observe that

$$u_\delta(0; b; f, g; \epsilon) \cap P_{\Omega_r}(B) = ((b; f, g)|_{P_{\Omega_r}(B) \cap S_\delta})^{-1}(D_\epsilon)$$

where we denote $D_\epsilon = \{z \in \mathrm{C}; |z| < \epsilon\}$. Let us define

$$Z_\delta := (P^* \pi_\omega P)^{-1}(S_\delta) = \{\zeta \in Z; \|P(\zeta)^* \pi_\omega(\zeta) P(\zeta)\| \le \delta\},$$

then

$$\begin{aligned} Z_\delta &= \{\zeta \in Z; \|P(\zeta)\|_r^2 \le \delta\} \\ &= \{\zeta \in Z; \sup\{\|x_i(\zeta)\|^2; (x_i)_{i=1}^{\infty} \subset \Omega \text{ is dense in } \|x\|_r \le 1\} \le \delta\}. \end{aligned}$$

Since $\|x_i(\zeta)\|$ is a μ -measurable function, we have $Z_\delta \in \mathcal{B}(Z)$. Then,

$$\begin{aligned} (P^* \pi_\omega P)^{-1}(u_\delta(0; b; f, g; \epsilon) \cap P_{\Omega_r}(B)) &= ((b; f, g)|_{P_{\Omega_r}(B) \cap S_\delta} \circ (P^* \pi_\omega P))^{-1}(D_\epsilon) \\ &= F(b; f, g)^{-1}(D_\epsilon) \cap Z_\delta \in \mathcal{B}(Z) \end{aligned}$$

since $F(b; f, g)$ is μ-measurable.

We next consider the general case $\phi \ne 0$. Observe that

$$u_\delta(\phi; b; f, g; \epsilon) = (\phi + u_\infty(0; b; f, g; \epsilon)) \cap S_\delta = g_\phi(u_\infty(0; b; f, g; \epsilon)) \cap S_\delta$$

where

$$u_\infty(0; b; f, g; \epsilon) := (b; f, g)^{-1}(D_\epsilon) = \{\varphi \in CP(B, B(\Omega_r)); |(\varphi(b) f, g)| < \epsilon\},$$

and $g_\phi : CB(B, B(\Omega_r)) \to CB(B, B(\Omega_r))$ is defined by

$$g_\phi : \varphi \mapsto \varphi + \phi \qquad \text{for } \varphi \in CB(B, B(\Omega_r)),$$

where $CB(B, B(\Omega_r)) = CP(B, B(\Omega_r)) - CP(B, B(\Omega_r))$ is the set of all completely bounded maps from B to $B(\Omega_r)$ (cf. e.g., [59]). Then,

$$
\begin{aligned}
&(P^*\pi_\omega P)^{-1}(u_\delta(\phi; b; f, g; \epsilon) \cap P_{\Omega_r}(B)) \\
&= (P^*\pi_\omega P)^{-1}(g_\phi(u_\infty(0; b; f, g; \epsilon)) \cap (P_{\Omega_r}(B) \cap S_\delta)) \\
&= (P^*\pi_\omega P)^{-1}(g_\phi(u_\infty(0; b; f, g; \epsilon)) \cap (P^*\pi_\omega P)^{-1}(P_{\Omega_r}(B) \cap S_\delta) \\
&= ((b; f, g) \circ g_{-\phi} \circ (P^*\pi_\omega P))^{-1}(D_\epsilon) \cap (P^*\pi_\omega P)^{-1}(u_\delta(0; b; f, g; \epsilon') \cap P_{\Omega_r}(B))
\end{aligned}
$$

where $\epsilon' > \delta\|b\|\|f\|\|g\|$. Note that

$$(b; f, g) \circ g_{-\phi} \circ (P^*\pi_\omega P) : \zeta \in Z \mapsto (P(\zeta)^*\pi_\omega(\zeta)(b)P(\zeta)f, g) - (\phi(b)f, g)$$

is a μ-measurable function of $\zeta \in Z$, hence we have

$$((b; f, g) \circ g_{-\phi} \circ (P^*\pi_\omega P))^{-1}(D_\epsilon) \in \mathcal{B}(Z),$$

and, from the first step, we have

$$(P^*\pi_\omega P)^{-1}(u_\delta(0; b; f, g; \epsilon') \cap P_\Omega(B)) \in \mathcal{B}(Z),$$

from which assertion (1) follows.

(2) : We have to show that the restriction map $r : \varphi \in P_{\Omega_r}(A) \mapsto \varphi|_B \in P_{\Omega_r}(B)$ is a BW-homeomorphism, i.e., r^{-1} is BW-continuous.

Let $F \subset P_{\Omega_r}(A)$ be BW-closed. Then, there exists a BW-closed set $\mathcal{C} \subset CP(A, B(\Omega_r))$ such that $F = \mathcal{C} \cap P_{\Omega_r}(A)$. For any $\delta > 0$, $F \cap S_\delta = (\mathcal{C} \cap S_\delta) \cap P_{\Omega_r}(A) = \mathcal{C}_\delta \cap P_{\Omega_r}(A)$ where $\mathcal{C}_\delta := \mathcal{C} \cap S_\delta$ is BW-closed. (Note that S_δ is BW-closed. cf.[11].) Since $r : P_{\Omega_r}(A) \to P_{\Omega_r}(B)$ is one to one and onto, we have

$$r(F \cap S_\delta) = r(\mathcal{C}_\delta \cap P_{\Omega_r}(A)) = r(\mathcal{C}_\delta) \cap r(P_{\Omega_r}(A)) = r(\mathcal{C}_\delta) \cap P_{\Omega_r}(B).$$

Now $\mathcal{C}_\delta$ is BW-compact (cf.[11]) and r is BW-continuous, so it follows that $r(\mathcal{C}_\delta)$ is BW-compact, hence BW-closed, so that $r(F \cap S_\delta)$ is BW-closed in the relative topology in $P_{\Omega_r}(B) \cap S_\delta$ induced from the BW-topology on $CP(B, B(\Omega_r)) \cap S_\delta$. Namely, we have proved that

$$r : P_{\Omega_r}(A) \cap S_\delta \to P_{\Omega_r}(B) \cap S_\delta$$

is a BW-homeomorphism for all $\delta > 0$. This implies that $r : P_{\Omega_r}(A) \to P_{\Omega_r}(B)$ is a BW-homeomorphism, so we proved (2), hence Lemma 9.6. ▮

Now we can define the extension of the pre-nuclear CP-map ψ by

$$\tilde{\psi} = \int_Z V(\zeta)^* \tilde{\pi}_\omega(\zeta) V(\zeta)\, d\mu(\zeta)$$

where $\tilde{\pi}_\omega(\zeta) = r^{-1}(\pi_\omega(\zeta))$. In particular, we can take $H = \Omega_r$ and $V = T_{\Omega_r} : \Omega_r \to H_{\pi_\omega}$, which is the continuous extension of the nuclear embedding $T : \Omega \to H_{\pi_\omega}$, and define $\pi_\omega^{\Omega_r} := T_{\Omega_r}^* \pi_\omega T_{\Omega_r}$. (Note that $\pi_\omega^{\Omega_r}$ is not a representation on Ω_r in general.) Then, for any $f, g \in \Omega$ and $a \in A$,

$$\begin{aligned}(\tilde{\pi}_\omega^{\Omega_r}(a) f, g)_r = \int_Z (T_{\Omega_r}(\zeta)^* \tilde{\pi}_\omega(\zeta)(a) T_{\Omega_r}(\zeta) f, g)_r \, d\mu(\zeta) \\ = \int_Z (\tilde{\pi}_\omega(\zeta)(a) f(\zeta), g(\zeta))\, d\mu(\zeta),\end{aligned}$$

where the suffix r denotes the scalar product in Ω_r. Since Ω can be an arbitrary countably Hilbert nuclear space such that $\Omega \subset H_{\pi_\omega} \subset \Omega'$ is a rigged Hilbert space, the above equality is true for any $f, g \in \Omega \subset H_{\pi_\omega}$, considering the unitary transforms on H_{π_ω}. Since the right hand side defines a bounded sesqui-linear form for $f, g \in H_{\pi_\omega}$, which does not depend on the choice of Ω, hence it defines an operator $\tilde{\pi}_\omega(a) \in$

$B(H_{\pi_\omega})$ (e.g., as an application of the Banach-Steinhaus theorem). We conclude, from the right hand side, that

$$\tilde{\pi}_\omega(a) = \int_Z^{\oplus} \tilde{\pi}_\omega(\zeta)(a)\, d\mu(\zeta) \qquad \text{for } a \in A.$$

This shows that $\tilde{\pi}_\omega$ is a representation of A on $H_{\tilde{\pi}_\omega} = H_{\pi_\omega}$ which extends π_ω.

We now come back to the proof of Proposition 9.5, namely $\tilde{\pi}_\omega(A)" = \pi_\omega(B)"$ for all $\omega \in S(B)$. To prove this, we will show

$$\tilde{\pi}_\omega(A)' \supset \pi_\omega(B)', \tag{4}$$

from which $\tilde{\pi}_\omega(A)" \subset \pi_\omega(B)"$ follows, and, since $\tilde{\pi}_\omega(A)" \supset \pi_\omega(B)"$ is obvious from $\tilde{\pi}_\omega(A) \supset \pi_\omega(B)$, the proof of (4) will complete the proof.

We use the following theorem proved by W.B. Arveson [11].

THEOREM 9.7 ([11; THEOREM 1.4.2]). *Let A be a unital C^*-algebra, H be a Hilbert space, and $\varphi \in CP(A, B(H))$ be represented as $\varphi = V^*\pi V$ where π is a representation of A on a Hilbert space K, and $V \in B(H, K)$. Then, there exists an affine order isomorphism from the partially ordered convex set of operators $(\pi(A)')_1^+ = \{T \in \pi(A)'; 0 \leq T \leq I_{H_\pi}\}$ onto the interval $[0, \varphi] = \{\psi \in CP(A, B(H); 0 \leq \psi \leq \varphi\}$ with the order induced by the cone $CP(A, B(H))$; this isomorphism is defined by*

$$(\pi(A)')_1^+ \ni T \leftrightarrow V^* T^{\frac{1}{2}} \pi T^{\frac{1}{2}} V \in [0, \varphi].$$

We apply the above theorem for $\varphi = \pi_\omega \in CP(B, B(H_{\pi_\omega}))$. By the above theorem, in order to prove (4), it suffices to show that there exists a injective embedding s from $[0, \pi_\omega]$ into $[0, \tilde{\pi}_\omega]$ such that $r \circ s = id$ on $[0, \pi_\omega]$.

We define this map $s : [0, \pi_\omega] \to [0, \tilde{\pi}_\omega]$ as follows. From the above Arveson's theorem, for every $\psi \in [0, \pi_\omega]$, there exists $T_\psi \in (\pi_\omega(B)')_1^+$ such that $\psi = T_\psi^{\frac{1}{2}} \pi_\omega T_\psi^{\frac{1}{2}}$. Let us define the extension of ψ by

$$\tilde{\psi} = T_\psi^{\frac{1}{2}} \tilde{\pi}_\omega T_\psi^{\frac{1}{2}}.$$

Note that this definition, made for general $\psi \in [0, \pi_\omega]$, coinsides, in the particular case where ψ is pre-nuclear, with the original definition one obtains from CP-decomposition. We then define s by

$$s(\psi) := \tilde{\psi} \qquad \text{for } \psi \in [0, \pi_\omega].$$

It is immediate to see that $(r \circ s)(\psi) = \psi$, since $(r \circ s)(\psi) = r(\tilde{\psi}) = r(T_\psi^{\frac{1}{2}} \tilde{\pi}_\omega T_\psi^{\frac{1}{2}}) = T_\psi^{\frac{1}{2}} r(\tilde{\pi}_\omega) T_\psi^{\frac{1}{2}} = T_\psi^{\frac{1}{2}} \pi_\omega T_\psi^{\frac{1}{2}} = \psi$. Hence s is injective.

To see that s is into, we have to show that $0 \leq s(\psi) \leq \tilde{\pi}_\omega$. Since $s(\psi) = \tilde{\psi} = T_\psi^{\frac{1}{2}} \tilde{\pi}_\omega T_\psi^{\frac{1}{2}} \in CP(A, B(H_{\pi_\omega}))$, $s(\psi) \geq 0$ is obvious, so we will show $s(\psi) \leq \tilde{\pi}_\omega$, i.e., $\tilde{\pi}_\omega - s(\psi) = \tilde{\pi}_\omega - T_\omega^{\frac{1}{2}} \tilde{\pi}_\omega T_\psi^{\frac{1}{2}} \in CP(A, B(H))$. To prove this, we observe that, since $T_\psi \in \pi_\omega(B)'$, we have

$$\pi_\omega = T_\psi^{\frac{1}{2}} \pi_\omega T_\psi^{\frac{1}{2}} + (I - T_\psi)^{\frac{1}{2}} \pi_\omega (I - T_\psi)^{\frac{1}{2}}.$$

So, if we prove

$$\tilde{\pi}_\omega = T_\psi^{\frac{1}{2}} \tilde{\pi}_\omega T_\psi^{\frac{1}{2}} + (I - T_\psi)^{\frac{1}{2}} \tilde{\pi}_\omega (I - T_\psi)^{\frac{1}{2}}, \tag{5}$$

then our assertion $s(\psi) \leq \tilde{\pi}_\omega$ follows, because

$$\tilde{\pi}_\omega - T_\psi^{\frac{1}{2}} \tilde{\pi}_\omega T_\psi^{\frac{1}{2}} = (I - T_\psi)^{\frac{1}{2}} \tilde{\pi}_\omega (I - T_\psi)^{\frac{1}{2}} \in CP(A, B(H_{\pi_\omega})).$$

Hence the proof of Theorem 9.1 is reduced to the proof of equality (5), which will be accomplished in several steps (Lemma 9.8 to Lemma 9.14).

One can show that (5) is equivalent to asserting that the map s is CP-affine, which is a CP-version of Effros' theorem [31; Th.11.1]. However, we will not focus on this fact in the subsequent arguments.

LEMMA 9.8. *Let* $V_1 = T_\psi^{\frac{1}{2}}$, $V_2 = (I - T_\psi)^{\frac{1}{2}}$ *and define*

$$K_\omega := [(\pi_\omega \oplus \pi_\omega)(B)(V_1 \oplus V_2)H_{\pi_\omega}]$$

$$\kappa_\omega := (\pi_\omega \oplus \pi_\omega)|_{K_\omega}$$

Then π_ω *and* κ_ω *are unitarily equivalent, i.e.,*

$$\pi_\omega = U_\omega^* \kappa_\omega U_\omega$$

where the unitary operator U_ω *is defined as*

$$U_\omega := V_1 \oplus V_2 : \xi \in H_{\pi_\omega} \mapsto V_1\xi \oplus V_2\xi \in H_{\pi_\omega} \oplus H_{\pi_\omega}.$$

PROOF: $V_1 \oplus V_2$ is a partial isometry from H_{π_ω} into $H_{\pi_\omega} \oplus H_{\pi_\omega}$, because for $\xi \in H_{\pi_\omega}$,

$$\begin{aligned}\|V_1\xi \oplus V_2\xi\|^2 &= (V_1\xi \oplus V_2\xi, V_1\xi \oplus V_2\xi) = (V_1\xi, V_1\xi) + (V_2\xi, V_2\xi) \\ &= (\xi, V_1^*V_1\xi) + (\xi, V_2^*V_2\xi) = (\xi, T_\psi\xi) + (\xi, (I - T_\psi)\xi) = (\xi, \xi) = \|\xi\|^2\end{aligned}$$

Noting that $V_1, V_2 \in \pi_\omega(B)'$ (since $T_\psi \in \pi_\omega(B)'$), we have

$$K_\omega = [(\pi_\omega \oplus \pi_\omega)(B)(V_1 \oplus V_2)H_{\pi_\omega}] = [(V_1 \oplus V_2)H_{\pi_\omega}] = (V_1 \oplus V_2)H_{\pi_\omega}$$

where the last equality holds since $V_1 \oplus V_2$ is a partial isometry; hence $U_\omega = V_1 \oplus V_2 : H_{\pi_\omega} \to K_\omega$ is a unitary operator from H_{π_ω} onto K_ω.

It is straightforward to show $\pi_\omega = U_\omega^* \kappa_\omega U_\omega$. In fact, for any $b \in B$ and $\xi \in H_{\pi_\omega}$,

$$\begin{aligned}(V_1 \oplus V_2)\pi_\omega(b)\xi &= V_1\pi_\omega(b)\xi \oplus V_2\pi_\omega(b)\xi \\ &= \pi_\omega(b)V_1\xi \oplus \pi_\omega(b)V_2\xi = (\pi_\omega(b) \oplus \pi_\omega(b))(V_1 \oplus V_2)\xi,\end{aligned}$$

since $V_1, V_2 \in \pi_\omega(B)'$. ∎

We denote by ξ_ω the cyclic vector of the cyclic representation π_ω, and define

$$D_\omega := \pi_\omega(B)\xi_\omega.$$

Note that D_ω is a dense subspace of H_{π_ω}, and stable under the action of $\pi_\omega(B)$. We also define $P_\omega(\zeta) : D_\omega \to H_{\pi_\omega}(\zeta)$ by

$$P_\omega(\zeta) : \pi_\omega(b)\xi_\omega \in D_\omega \mapsto \pi_\omega(\zeta)(b)\xi_\omega(\zeta) \in H_{\pi_\omega}(\zeta),$$

where $b \in B$. We also note that $P_\omega(\zeta)$ is μ- a.e. defined, which is enough for our purpose to formulate a unitary equivalence of irreducible representations as integrands (cf. Lemma 9.12 and Lemma 9.13).

We denote

$$D_\omega(\zeta) := P_\omega(\zeta)D_\omega$$

and

$$N_\omega(\zeta) := ker\, P_\omega(\zeta).$$

We note that $N_\omega(\zeta) = \pi_\omega(I_\zeta)\xi_\omega$, where I_ζ is the left ideal of B defined by

$$I_\zeta = \{b \in B; \pi_\omega(\zeta)(b)\xi_\omega(\zeta) = 0\}.$$

Furthermore, for the simplification of our subsequent arguments, we introduce the following notations;

$$\tau(\zeta) := T_\psi^{\frac{1}{2}}(\zeta) \oplus (I - T_\psi)^{\frac{1}{2}}(\zeta) : D_\omega \to H_{\pi_\omega}(\zeta) \oplus H_{\pi_\omega}(\zeta)$$

$$\text{where} \quad \tau(\zeta)\xi = (T_\psi^{\frac{1}{2}}\xi)(\zeta) \oplus ((I - T_\psi)^{\frac{1}{2}}\xi)(\zeta) \quad \text{for} \quad \xi \in D_\omega, \quad \text{and}$$

$$\underline{K}_\omega(\zeta) := \tau(\zeta)D_\omega$$

$$\underline{\kappa}_\omega(\zeta) := (\pi_\omega(\zeta) \oplus \pi_\omega(\zeta))|_{\underline{K}_\omega(\zeta)}$$

$$\hat{K}_\omega(\zeta) := \tau(\zeta)D_\omega/\tau(\zeta)N_\omega(\zeta)$$

and we denote by $\iota(\zeta)$ the canonical map $\iota(\zeta) : \tau(\zeta)D\omega \to \hat{K}_\omega(\zeta)$. In order to define a representation on $\hat{K}_\omega$ induced from $\underline{\kappa}_\omega$, we need to prove the following lemma.

LEMMA 9.9. *$\tau(\zeta)N_\omega(\zeta)$ is stable under $\underline{\kappa}_\omega(\zeta)$, i.e.,*

$$\underline{\kappa}_\omega(\zeta)(B)(\tau(\zeta)N_\omega(\zeta)) \subset \tau(\zeta)N_\omega(\zeta).$$

PROOF: We first note that, from Lemma 9.8, $\kappa_\omega U_\omega = U_\omega \pi_\omega$, i.e.,

$$(\pi_\omega \oplus \pi_\omega)(T_\psi^{\frac{1}{2}} \oplus (I - T_\psi)^{\frac{1}{2}}) = (T_\psi^{\frac{1}{2}} \oplus (I - T_\psi^{\frac{1}{2}}))\pi_\omega.$$

Let $b \in B$ and $\xi \in D_\omega$ be arbitrary. Then,

$$[(\pi_\omega(b) \oplus \pi_\omega(b))(T_\psi^{\frac{1}{2}} \oplus (I - T_\psi)^{\frac{1}{2}})\xi](\zeta) = [(T_\psi^{\frac{1}{2}} \oplus (I - T_\psi)^{\frac{1}{2}})\pi_\omega(b)\xi](\zeta),$$

so that

$$(\pi_\omega(\zeta)(b) \oplus \pi_\omega(\zeta)(b))(T_\psi^{\frac{1}{2}}(\zeta) \oplus (I - T_\psi)^{\frac{1}{2}}(\zeta))\xi = (T_\psi^{\frac{1}{2}}(\zeta) \oplus (I - T_\psi)^{\frac{1}{2}}(\zeta))\pi_\omega(b)\xi,$$

i.e.,

$$\underline{\kappa}_\omega(\zeta)(b)\tau(\zeta)\xi = \tau(\zeta)\pi_\omega(b)\xi. \tag{6}$$

We next note that, if $\xi \in N_\omega(\zeta)$, then $\pi_\omega(b)\xi \in N_\omega(\zeta)$ for all $b \in B$, i.e.,

$$\pi_\omega(B)\, N_\omega(\zeta) \subset N_\omega(\zeta).$$

In fact,

$$\pi_\omega(B) N_\omega(\zeta) = \pi_\omega(B)\pi_\omega(I_\zeta)\xi_\omega \subset \pi_\omega(I_\zeta)\xi_\omega = N_\omega(\zeta).$$

From (6), it follows that

$$\underline{\kappa}_\omega(\zeta)(B)(\tau(\zeta)\, N_\omega(\zeta)) = \tau(\zeta)\pi_\omega(B)\, N_\omega(\zeta) \subset \tau(\zeta)\, N_\omega(\zeta).$$

∎

From Lemma 9.9, $\xi_1, \xi_2 \in \tau(\zeta)D_\omega = \underline{K}_\omega(\zeta)$ with $\xi_1 \sim \xi_2 \pmod{\tau(\zeta)\, N_\omega(\zeta)}$ implies $\underline{\kappa}_\omega(\zeta)(b)\xi_1 \sim \underline{\kappa}_\omega(\zeta)(b)\xi_2 \pmod{\tau(\zeta)\, N_\omega(\zeta)}$ for all $b \in B$. Thus, we can define a representation $\hat{\kappa}_\omega(\zeta)$ of B acting on the linear space $\hat{K}_\omega(\zeta)$, induced by $\underline{\kappa}_\omega$;

$$\hat{\kappa}_\omega(\zeta)(b)(\xi + \tau(\zeta)\, N_\omega(\zeta)) := \underline{\kappa}_\omega(\zeta)(b)\xi + \tau(\zeta)\, N_\omega(\zeta),$$

or symbolically,

$$\hat{\kappa}_\omega(b)(\zeta)(\iota(\zeta)\xi) := \iota(\zeta)(\underline{\kappa}_\omega(\zeta)(b)\xi)$$

for all $b \in B$ and $\xi \in \underline{K}_\omega(\zeta)$.

LEMMA 9.10. *There exists an isomorphism*

$$\hat{U}_\omega(\zeta) : D_\omega(\zeta) \to \hat{K}_\omega(\zeta).$$

PROOF: We consider the following diagram:

$$\begin{tikzcd}
D_\omega \arrow[r, "\tau(\zeta)"] \arrow[d, "P_\omega(\zeta)"'] & \underline{K}_\omega(\zeta) \arrow[d, "\iota(\zeta)"] \\
D_\omega(\zeta) \arrow[r, "\hat{U}_\omega(\zeta)"] & \hat{K}_\omega(\zeta)
\end{tikzcd}$$

Here we define $\hat{U}_\omega(\zeta)$ on $D_\omega(\zeta)$ by

$$\hat{U}_\omega(\zeta) : P_\omega(\zeta)\xi \mapsto \iota(\zeta)\ \tau(\zeta)\xi \qquad \text{for all } \xi \in D_\omega.$$

Then $\hat{U}_\omega(\zeta)$ is well defined, bijective and continuous since $ker(\iota(\zeta) \circ \tau(\zeta)) = ker\ P_\omega(\zeta) = N_\omega(\zeta)$, i.e., we have the isomorphism

$$D_\omega(\zeta) \cong D_\omega / N_\omega(\zeta) = D_\omega \Big/ ker(\iota(\zeta) \circ \tau(\zeta)) \cong \hat{K}_\omega(\zeta).$$

∎

LEMMA 9.11. *$D_\omega(\zeta)$ is norm dense in $H_{\pi_\omega}(\zeta)$ for μ- a.e. $\zeta \in Z$.*

PROOF: From the definition of measurable fields of Hilbert spaces, there exists a countable family $(\xi_i) \subset H_{\pi_\omega}$ such that, for every $\zeta \in Z$, $(\xi_i(\zeta))$ is total in $H_{\pi_\omega}(\zeta)$. Since D_ω is norm dense in H_{π_ω}, we can take a sequence $(\xi_j^{(i)}) \subset D_\omega$ which converges to ξ_i in L^2-norm for each $i \in \mathbb{N}$. Then, from [29; II, Ch.1, Prop. 5], for each fixed i, we can take a subsequence $(\xi_{j_k}^{(i)})$ of $(\xi_j^{(i)})$ such that $(\xi_{j_k}^{(i)})$ converges to $\xi_i(\zeta)$ in H_{π_ω} for μ - a.e. $\zeta \in Z$, i.e., for $\zeta \in Z \setminus N_i$ with $\mu(N_i) = 0$. Let $N = \cup_{i=1}^{\infty} N_i$, then $(\xi_{j_k}^{(i)}(\zeta)) \to (\xi_i(\zeta))$ in $H_{\pi_\omega}(\zeta)$ for all $i \in \mathbb{N}$ and $\zeta \in Z \setminus N$, where $\mu(N) \leq \sum_{i=1}^{\infty} \mu(N_i) = 0$. Noting that the double sequence $(\xi_{j_k}^{(i)}) \subset D_\omega$ is countable, and $(\xi_i(\zeta))$ is total in H_{π_ω} for every $\zeta \in Z$, we conclude that $(\xi_{j_k}^{(i)}(\zeta))$ is total in $H_{\pi_\omega}(\zeta)$ for $\zeta \in Z \setminus N$. Hence, $D_\omega(\zeta)$ is norm dense in $H_{\pi_\omega}(\zeta)$ for μ- a.e. $\zeta \in Z$. ∎

By Lemma 9.10 and Lemma 9.11, we can define the closure of $\hat{K}_\omega(\zeta)$, which is isomorphic to $H_{\pi_\omega}(\zeta)$ by a unitary operator $\bar{U}_\omega(\zeta)$ which is the continuous extension of $\hat{U}_\omega(\zeta)$. We then define a representation $\bar{\kappa}_\omega(\zeta)$ to be the continuous extension of $\hat{\kappa}_\omega(\zeta)$ to $\bar{K}_\omega(\zeta)$.

LEMMA 9.12. *The unitary operator $\bar{U}_\omega(\zeta)$ establishes a unitary equivalence between $\pi_\omega(\zeta)$ and $\bar{\kappa}_\omega(\zeta)$ for μ- a.e. $\zeta \in Z$, i.e.,*

$$\pi_\omega(\zeta) = \bar{U}_\omega(\zeta)^* \bar{\kappa}_\omega(\zeta) \bar{U}_\omega(\zeta)$$

PROOF: Observe first that, for any $b \in B$ and $\xi \in D_\omega$,

$$\hat{\kappa}_\omega(\zeta)(b)\hat{U}_\omega(\zeta)(P_\omega(\zeta)\xi)$$

$$= \hat{\kappa}_\omega(\zeta)(b)(\iota(\zeta) \circ \tau(\zeta))\xi \qquad \text{(cf. diagram of Lemma 9.10)}$$

$$= \iota(\zeta)(\underline{\kappa}_\omega(\zeta)(b)\tau(\zeta)\xi) \qquad \text{(cf. definition of } \hat{\kappa}_\omega(\zeta))$$

$$= \iota(\zeta)(\tau(\zeta)\pi_\omega(b)\xi) \qquad \text{(cf. (6) in the proof of Lemma 9.9)}$$

$$= \hat{U}_\omega(\zeta) \circ P_\omega(\zeta)(\pi_\omega(b)\xi) \qquad \text{(cf. diagram of Lemma 9.10)}$$

$$= \hat{U}_\omega(\zeta)\pi_\omega(\zeta)(b)(P_\omega(\zeta)\xi) \qquad \text{(cf. definition of } P_\omega(\zeta)).$$

Since $D_\omega(\zeta) = P_\omega(\zeta)D_\omega$ is norm dense in $H_{\pi_\omega}(\zeta)$ for μ- a.e. $\zeta \in Z$ (Lemma 9.11), the continuous extension of the above equality proves our assertion. ∎

Lemma 9.13. *Let us define*

$$\bar{K}_\omega = \int_Z^{\oplus} \bar{K}_\omega(\zeta)\, d\mu(\zeta)$$

$$\bar{\kappa}_\omega = \int_Z^{\oplus} \bar{\kappa}_\omega(\zeta)\, d\mu(\zeta)$$

$$\bar{U}_\omega = \int_Z^{\oplus} \bar{U}_\omega(\zeta)\, d\mu(\zeta)$$

Then,

$$\bar{K}_\omega = \bar{U}_\omega H_{\pi_\omega} \quad \text{and} \quad \pi_\omega = \bar{U}_\omega^* \bar{\kappa}_\omega \bar{U}_\omega,$$

and we can identify

$$K_\omega = \bar{K}_\omega, \quad \kappa_\omega = \bar{\kappa}_\omega \quad \text{and} \quad U_\omega = \bar{U}_\omega$$

in the sense of unitary equivalence.

Proof: The equalities $\bar{K}_\omega = \bar{U}_\omega H_{\pi_\omega}$ and $\pi_\omega = \bar{U}_\omega^* \bar{\kappa}_\omega \bar{U}_\omega$ follow imediately from Lemma 9.12. Since, by Lemma 9.8,

$$(\pi_\omega, H_{\pi_\omega}) = U_\omega^* (\kappa_\omega, K_\omega) U_\omega$$

and π_ω has an irreducible decomposition

$$(\pi_\omega, H_{\pi_\omega}) = \int_Z^{\oplus} (\pi_\omega(\zeta), H_{\pi_\omega}(\zeta))\, d\mu(\zeta),$$

according to [29; II, Ch.3, Prop. 11], there exists an irreducible decomposition of $(\kappa_\omega . K_\omega)$, e.g.,

$$(\kappa_\omega, K_\omega) = \int_Z^{\oplus} (\check{\kappa}_\omega(\zeta), \check{K}_\omega(\zeta))\, d\mu(\zeta)$$

and a measurable field of unitary operators

$$\check{U}_\omega(\zeta) : H_{\pi_\omega}(\zeta) \to \check{K}_\omega(\zeta)$$

such that

$$(\pi_\omega(\zeta), H_{\pi_\omega}(\zeta)) = \breve{U}_\omega(\zeta)^*(\breve{\kappa}_\omega(\zeta), \breve{K}_\omega(\zeta))\breve{U}_\omega(\zeta).$$

On the other hand, we have just proved

$$(\pi_\omega, H_{\pi_\omega}) = \bar{U}_\omega^*(\bar{\kappa}_\omega, \bar{K}_\omega)\bar{U}_\omega$$

and

$$(\bar{\kappa}_\omega, \bar{K}_\omega) = \int_Z^\oplus (\bar{\kappa}_\omega(\zeta), \bar{K}_\omega(\zeta))\, d\mu(\zeta)$$

with

$$(\pi_\omega(\zeta), H_{\pi_\omega}(\zeta)) = \bar{U}_\omega(\zeta)^*(\bar{\kappa}_\omega(\zeta), \bar{K}_\omega(\zeta))\bar{U}_\omega(\zeta).$$

Noting that both $(\kappa_\omega, K_\omega)$ and $(\bar{\kappa}_\omega, \bar{K}_\omega)$ are subrepresentations of B in $H_{\pi_\omega} \oplus H_{\pi_\omega}$ which are unitarily equivalent to $(\pi_\omega, H_{\pi_\omega})$, we conclude that, by the uniqueness of the irreducible decomposition, we can identify $(\kappa_\omega, K_\omega)$ with $(\bar{\kappa}_\omega, \bar{K}_\omega)$, and U_ω with $\bar{U}_\omega$ in the sense of unitary equivalence. ∎

LEMMA 9.14.

$$\tilde{\pi}_\omega = U_\omega^* \tilde{\kappa}_\omega U_\omega$$

where

$$\tilde{\kappa}_\omega = (\tilde{\pi}_\omega \oplus \tilde{\pi}_\omega)|_{K_\omega} \qquad \textit{(on } A\textit{)}.$$

PROOF: From Lemma 9.13, we can set

$$\kappa_\omega(\zeta) := \bar{\kappa}_\omega(\zeta), \quad K_\omega(\zeta) := \bar{K}_\omega(\zeta) \quad \text{and} \quad U_\omega(\zeta) := \bar{U}_\omega(\zeta).$$

Then, from Lemma 9.12, we have

$$\pi_\omega(\zeta) = U_\omega(\zeta)^* \kappa_\omega(\zeta) U_\omega(\zeta) \qquad (\text{on } B).$$

By Theorem 9.2, B separates $Irr(A)$, so that, since $\pi_\omega(\zeta)$ and $\kappa_\omega(\zeta)$ are unitarily equivalent with the unitary operator $U_\omega(\zeta)$ (i.e., $\pi_\omega(\zeta) = U_\omega(\zeta)^*\kappa_\omega(\zeta)U_\omega(\zeta)$), $\tilde{\pi}_\omega(\zeta)$ and $\tilde{\kappa}_\omega(\zeta)$ are unitarily equivalent with the same unitary operator $U_\omega(\zeta)$, i.e.,

$$\tilde{\pi}_\omega(\zeta) = U_\omega(\zeta)^*\tilde{\kappa}_\omega(\zeta)U_\omega(\zeta) \qquad \text{(on } A).$$

On the other hand, recalling the definition of $\kappa_\omega(\zeta)$, i.e.,

$$\kappa_\omega(\zeta) = (\pi_\omega(\zeta) \oplus \pi_\omega(\zeta))|_{K_\omega(\zeta)} \qquad \text{(on } B),$$

and noting that, as we showed above, the unitary equivalence $\tilde{\pi}_\omega(\zeta) \simeq \tilde{\kappa}_\omega(\zeta)$ is given by the same unitary operator $U_\omega(\zeta)$ for the unitary equivalence $\pi_\omega(\zeta) \simeq \kappa_\omega(\zeta)$, and $\kappa_\omega(\zeta) \in Irr(B)$ has a unique extension to $Irr(A)$ (cf. Theorem 9.2)), we conclude that

$$\tilde{\kappa}_\omega(\zeta) = (\tilde{\pi}_\omega(\zeta) \oplus \tilde{\pi}_\omega(\zeta)|_{K_\omega(\zeta)} \qquad \text{(on } A).$$

Integrating these expressions, we have

$$\tilde{\pi}_\omega = \int_Z^\oplus U_\omega(\zeta)^*\tilde{\kappa}_\omega(\zeta)U_\omega(\zeta)\, d\mu(\zeta) = U_\omega^*\tilde{\kappa}_\omega U_\omega$$

and

$$\tilde{\kappa}_\omega = \int_Z^\oplus (\tilde{\pi}_\omega(\zeta) \oplus \tilde{\pi}_\omega(\zeta))|_{K_{\omega(\zeta)}}\, d\mu(\zeta) = (\tilde{\pi}_\omega \oplus \tilde{\pi}_\omega)|_{K_\omega}.$$

∎

We can now complete the proof of Theorem 9.1. Recall that it was reduced to showing

(5) $$\tilde{\pi}_\omega = V_1^*\tilde{\pi}_\omega V_1 + V_2^*\tilde{\pi}_\omega V_2$$

where $V_1 = T_\psi^{\frac{1}{2}}$, $V_2 = (I - T_\psi)^{\frac{1}{2}}$. Now, this follows immediately from Lemma 9.14, i.e.,

$$\begin{aligned}\tilde{\pi}_\omega &= U_\omega^* \tilde{\kappa}_\omega U_\omega \\ &= (V_1 \oplus V_2)^* (\tilde{\pi}_\omega \oplus \tilde{\pi}_\omega)(V_1 \oplus V_2) \\ &= V_1^* \tilde{\pi}_\omega V_1 + V_2^* \tilde{\pi}_\omega V_2.\end{aligned}$$

This completes the proof of Theorem 9.1. ∎

APPENDIX

The purpose of this appendix is to summarize the definitions and basic properties on operator algebras, completely positive maps, scalar convexity theory and nuclear spaces which we are based on in this dissertation. These are not intended as a comprehensive survey of the subjects since the detailed expository descriptions are available in many references, and the definitions of more technical terms and related results were explained in the main text.

1. C*-and W*-algebras.

A C*-algebra A is defined to be a Banach *-algebra such that $\|a^*a\| = \|a\|^2$ for all $a \in A$. A W*-algebra M is a C*-algebra which has a predual, i.e., there exists a Banach space M_* such that $M = (M_*)^*$. For example, every commutative C*-algebra is isomorphic to $C_{\mathbb{C}}(X) :=$ {$\mathbb{C}$-valued continuous functions on a compact Hausdorff space X}, and every non-commutative C*-algebra is realized on a Hilbert space H, e.g., $C(H) :=$ {compact operators on H} is a C*-algebra, and $B(H) :=$ {bounded linear operators on H} is a W*-algebra with the predual $B(H)_* = \mathcal{T}(H)$ $:=$ {trace class operators on H}. (See references for other examples.)

The self-adjoint part A_{sa} of a C*-algebra A is partially ordered by the positive cone $A^+ = \{a^*a\,;\, a \in A\} = \{a^2\,;\, a \in A_{sa}\}$. A positive linear functional $\omega \in (A^*)^+$

is called a ***state*** if $\|\omega\| = 1$, and a ***quasi-state*** if $\|\omega\| \leq 1$. $\omega \in (A*)^+$ is defined to be ***pure*** if $0 \leq \eta \leq \omega$ with $\eta \in (A^*)^+$ implies $\eta = c\omega$ with $0 \leq c \leq 1$. We denote by $S(A)$ [resp. $Q(A), \tilde{P}(A), P(A)$] the set of all states [resp. quasi-states, pure elements, pure states]. Similarly, for a W*-algebra M, we can define the ***normal state space*** [resp. ***normal quasi-state space***] by $N(M) := S(M) \cap M_*$ [resp. $Q(M)_n := Q(M) \cap M_*$].

A ***representation*** π of a C*-algebra A on a Hilbert space H is defined to be a *-homomorphism $\pi : A \to B(H)$. π is called *cyclic* if there exists $\xi \in H$ such that $[\pi(A)\xi] = H$, then ξ is called a ***cyclic vector*** for $\pi(A)$. For C*-algebras, the notions of topological and algebraic irreducibility coincide, i.e., π is *irreducible* if and only if $\{0\}$ and H are the only stable closed subspaces under the action of $\pi(A)$, or equivalently only the scalar operators can commute with all elements of $\pi(A)$. The irreducibility is also characterized by the property that every $\xi \in H$ is a cyclic vector for $\pi(A)$. We denote by $Rep(A)$ [resp. $Rep_c(A)$, $Irr(A)$] the set of all representations [resp. cyclic representations, irreducible representations] of A. If we specify the representation space H, we write $Rep(A : H)$ [resp. $Rep_c(A : H)$, $Irr(A : H)$].

By the GNS (Gelfand-Naimark-Segal) construction, for any quasi-state $\omega \in Q(A)$, there exists a cyclic representation π_ω of A on a Hilbert space H_ω and a cyclic vector $\xi_\omega \in H_\omega$ with $\|\omega\| = \|\xi_\omega\|^2$ such that

$$\omega(a) = (\pi_\omega(a)\xi_\omega, \xi_\omega) \quad \text{for all } a \in A.$$

Then, ω is pure if and only if π_ω is irreducible. The same arguments are applied for a normal quasi-state $\omega \in Q(M)_n$ of a W*-algebra M, where the GNS-representation

π_ω is normal, i.e., $\sigma(M, M_*) - \sigma(B(H_\omega), B(H_\omega)_*)$ continuous.

The second dual A^{**} of a C*-algebra A has a natural structure of a W*-algebra, which is called the ***enveloping W*-algebra*** of A. It is also characterized by the property that the embedding $A \hookrightarrow A^{**}$ is canonical, i.e., every representation $\pi : A \to B(H)$ is extended uniquely to a normal representation $\tilde{\pi} : A^{**} \to B(H)$. We can identify the quasi-state space $Q(A)$ of A with the normal quasi-state space $Q(A^{**})_n$ of A^{**}. Then, Kadison's function representation theorem states that

$$A \cong A_0(Q(A), \mathrm{C}) \qquad \text{and} \qquad A^{**} \cong A_0^b(Q(A), \mathrm{C})$$

where $A_0(Q(A), \mathrm{C})$ [resp. $A_0^b(Q(A), \mathrm{C})$] denotes the set of all C-valued w*-continu ous [resp. bounded] affine functions on $Q(A)$ which vanish at 0. The last isomorphism is generalized for a W*-algebra M as $M \cong A_0^b(Q(M)_n, \mathrm{C})$.

If A is a separable C*-algebra, any representation π of A on a separable Hilbert space H is decomposed into a direct integral of irreducible representations, i.e., for each maximal abelian subalgebra $\mathcal{M} \subset \pi(A)'$ (cf. section 6 below), there exists a standard measure space (Z, μ) corresponding to $\mathcal{M}$, and a measurable family $(\pi(\zeta))_{\zeta \in Z}$ of irreducible representations of A on $(H(\zeta))_{\zeta \in Z}$ such that

$$\pi = \int_Z^\oplus \pi(\zeta)\, d\mu(\zeta) \qquad \text{and} \qquad H = \int_Z^\oplus H(\zeta)\, d\mu(\zeta).$$

Finally, we mention about the tensor product of C*-algebras. The algebraic tensor product $A_1 \otimes A_2$ of C*-algebras A_i $(i = 1, 2)$ is a *-algebra, in which we can define C*-norms $\|\cdot\|_\alpha$ such that $\|\cdot\|_{min} \le \|\cdot\|_\alpha \le \|\cdot\|_{max}$, where

$$\|x\|_{min} = \sup\{\|(\pi_1 \otimes \pi_2)(x)\|; \pi_i \in Rep(A_i)\,(i = 1, 2)\}$$

$$\|x\|_{max} = \sup\{\|\pi(x)\|; \pi \in Rep(A_1 \otimes A_2)\}.$$

(cf. references [19], [28], [29], [52], [53], [60], [65], [72], [75])

2. Completely positive maps.

A linear map $\psi : A \to B$ between C*-algebras A and B is defined to be *completely positive* [resp. *completely bounded*], if, for every integer $n \geq 1$, the corresponding linear map

$$\psi_n : [a_{ij}] \in A \otimes M_n(\mathbb{C}) \to [\psi(a_{ij})] \in B \otimes M_n(\mathbb{C})$$

is positive for all n [resp. $\sup_n \|\psi_n\| < \infty$], where $M_n(\mathbb{C})$ denotes the C*-algebra of all complex $n \times n$ matrices. (For brevity, we then say that ψ is a *CP-map* [resp. *CB-map*].)

We denote by $CP(A, B(H))$ the set of all completely positive linear maps from a C*-algebra A into $B(H)$, then $CB(A, B(H)) = CP(A, B(H)) - CP(A, B(H))$. By the Stinespring representation theorem, every $\psi \in CP(A, B(H))$ can be written as

$$\psi(a) = V^*\pi(a)V \qquad \text{for all } a \in A,$$

where π is a representation of A on a Hilbert space K and $V \in B(H, K)$ is a bounded linear operator from H to K such that $\|\psi\| = \|V\|^2$; this is proved for the unital case in many references, but it is easily proved for the non-unital case using the approximate identity. Under the minimal condition : $K = [\pi(A)VH]$, the above representation is unique up to unitary intertwining operators. Complete positivity induces a partially ordering in the cone $CP(A, B(H))$, and we denote by

$P_H(A)$ the set of all pure elements of $CP(A, B(H))$ with this completely positive order. Then $\psi = V^*\pi V \in P_H(A)$ if and only if $\pi \in Irr(A)$. For a W*-algebra M, we denote the normal part of $CP(M, B(H))$ by $CP(M, B(H))_n$.

For a bounded subset of $CP(A, B(H))$, we define a weak topology, which is called *BW-topology,* as follows; a bounded net $(\psi_\alpha) \subset CP(A, B(H))$ converges to $\psi \in CP(A, B(H))$ in BW-topology if and only if $\psi_\alpha(a)$ converges weakly to $\psi(a)$ in $B(H)$ for all $a \in A$. It is shown in [11] that the BW-topology is a w*-topology as a Banach dual space, hence every bounded ball of $CP(A, B(H))$ is BW-compact.

A unital selfadjoint subspace $S \subset A$ of a unital C*-algebra A is called an *operator system.* Let $S \subset A$ be an operator system and B be a C*-algebra, then we define a map $\psi : S \to B$ to be *completely positive* with the same definition as for C*-algebras, i.e., the corresponding linear map $\psi_n : S \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C})$ is positive for all n. Then a C*-algebra A is called *injective,* if for every operator system $S \subset A$ and C*-algebra B, every completely positive map $\psi : S \to B$ can be extended on all of A. Arveson's extension theorem [11] shows that $B(H)$ is injective.

(cf. references [11], [59], [71], [76])

3. JB-and JBW-algebras.

A Jordan algebra is a non-associative algebra with $(a^2 \circ b) \circ a = a^2 \circ (b \circ a)$. A JB-algebra A is defined to be a Jordan Banach algebra over reals such that $\|a^2\| = \|a\|^2 \leq \|a^2 + b^2\|$ for all $a, b \in A$. A JBW-algebra M is a JB-algebra which has a predual M_*, i.e., $M = (M_*)^*$. For example, the self-adjoint part of any C*- [resp. W*-] algebra is a JB- [resp. JBW-] algebra with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. A Jordan algebra which is isomorphic to a norm [resp. weakly] closed Jordan subalgebra of $B(H)_{sa}$ (where H is a complex Hilbert space) is called a *JC-* [resp. *JW-*] *algebra,* which are generally called *special* Jordan algebras. A JB-algebra A is called *purely exceptional* if there is no non-zero homomorphism of A into a JC-algebra, i.e., every factor representation (see the definition below) of A is onto M_3^8 (:= the set of all self-adjoint 3×3 matrices over the Carley numbers). It is crucial that any JB-algebra A contains a unique purely exceptional ideal J such that A/J is a JC-algebra, and any JBW-algebra is uniquely decomposed into the special and purely exceptional parts.

A JB-algebra A is partially ordered by the positive cone $A^+ = \{a^2\,;\, a \in A\}$. Using the same definition as in the C*-algebra case, we can define the *state space* $S(A) := \{\omega \in (A^*)^+;\ \|\omega\| = 1\}$, the *quasi-state space* $Q(A) := \{\omega \in (A^*)^+;\ \omega \leq 1\}$, the set of *pure* elements $\tilde{P}(A)$, and the *pure states* $P(A) = \tilde{P}(A) \cap S(A)$; similarly, for a JBW-algebra M, we can define the *normal state space* $N(M) := S(M) \cap M_*$, and the *normal quasi-state space* $Q(M)_n := Q(M) \cap M_*$. For JB- or JW-algebras, Kadison's function representation theorem takes the form

$$A \cong A_0(Q(A), \mathbf{R}) \qquad \text{(order isomorphism)}$$

$$M \cong A_0^b(Q(M)_n, \mathbf{R}) \qquad \text{(order isomorphism)}$$

Any two elements a, b in a Jordan algebra A are said to *operator commute* if the multiplication operators $T_a : c \mapsto a \circ c$ and $T_b : c \mapsto b \circ c$ (for $c \in A$) commute, i.e., $(a \circ c) \circ b = a \circ (c \circ b)$ for all $c \in A$. The *center* of A is defined as the set of all elements of A which operator commute with every other elements of A. A JBW-algebra is called a *factor* if it contains no central idempotent elements other than 0 and the identity e. A JBW-factor is defined to be of *type I* if it contains a minimal idempotent. Then, a *representation* of a JB-algebra A is a homomorphism $\rho : A \to M$ from A into a type I JBW-factor M. ρ is said to be *dense* if $\overline{\rho(A)}^{w} = M$, i.e., $\rho(A)$ is a weakly dense in M. A JB-algebra A is defined to be of *complex type* if all dense representations $\pi : A \to B(H)_{sa}$ are onto a type I factor isomorphic to $B(H)_{sa}$ where H is a complex Hilbert space. Every dense representation is irreducible, however the converese is true if and only if A is of complex type.

A Jordan algebra $A \subset B(H)_{sa}$ is called *reversible* if $a_1, \ldots, a_n \in A$, then $a_1 \cdots a_n + a_n \cdots a_1 \in A$. A JC-algebra A is called *universally reversible* if $\pi(A)$ is reversible for any Jordan representation $\pi : A \to B(H)_{sa}$ (cf.[4],[44]). A JC-algebra is universally reversible if and only if it has no spin factor (or equivalently type I_2 facor) representation other than onto the spin factors $M_2(\mathbf{R})_s$ and $M_2(\mathbf{C})_{sa}$ (see [44;Th.2.2]). We also define a JW-algebra to be *universally reversible* if it is reversible for any normal Jordan representations.

Finally, we note that the double commutant A^{**} of a JB-algebra A has a natural structure of a JBW-algebra, which is called the *enveloping JBW-algebra* of A. Similarly as in the C*-algebra case, it is also characterized by the canonical property.

(cf. references [4], [5], [8], [44], [45], [67])

4. Convexity theory.

A *convex* set K is a set which is closed under the *convex combination* $\lambda x + \mu y$ with $x, y \in K$ and $\lambda, \mu > 0$ such that $\lambda + \mu = 1$. For any subset $S \subset K$, we define $conv.\, S := \{\lambda x + \mu y\,;\, x, y \in S$ and $\lambda, \mu > 0$ such that $\lambda + \mu = 1\}$. A subset $F \subset K$ is said to be a *face* of K if, for any $x, y \in K$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$, $\lambda x + \mu y \in F$ implies $x, y \in F$. A face consisting of a single element is called an *extreme point* . We denote the set of all extreme points of K by $\partial_e K$. A face F of K is called a *split face* of K, if there exists a face F' of K such that every element $x \in K$ is uniquely decomposed as a convex conbination of elements of F and F'.

For a subset S in a topological vector space, we denote by *cl.conv.* S the closure of *conv.* S. The Krein-Milman theorem states that, if K is a compact convex subset of a Hausdorff locally convex topological vector space, then $K = cl.conv.\, \partial_e K$. The Choquet-Bishop-de Leeuw theorem refines this theorem and states that every point in a compact convex set K is the barycenter of a probability measure which is pseudo-supported by the extreme points $\partial_e K$, i.e., it vanishes on every Baire subset of K disjoint from $\partial_e K$.

An ordered linear space is called *Archimedian* if $na \leq e$, with $n \in \mathbb{N}, a \in A$ implies $a \leq 0$. An *order unit space* (A, e) is an Archimedian ordered linear space such that A is generated by an order unit e, i.e., $A = \cup_{n \in \mathbb{N}} n[-e, e]$. An order unit space A is considered as a normed linear space equipped with the order unit norm $\|a\| = \inf\{\lambda \geq 0\,;\, a \in \lambda[-e, e]\}$. For example, a unital JB-algebra, hence as a particular case the self-adjoint part of a unital C*-algebra, is an order unit space. A *base norm space* (V, K) is defined to be a positively generated ordered linear space (i.e., $V = V^+ - V^+$) such that $K = \{\rho \in V^+;\ a(\rho) = 1\}$ for some $a \in (V^*)^+$ $(a \neq 0)$ and $B = conv\,(K \cup -K)$ is a radially bounded set, i.e., $B \cap l$ is a bounded segment for every line l through the origin of V. V is considered as a normed linear space with the base norm $\|\rho\| = \inf\{\lambda \geq 0;\ \rho \in \lambda B\}$. For example, the dual space of a unital JB-algebra (A, e) is a base norm space with the state space $K = S(A)$ as its base. More generally, order unit spaces and base norm spaces are the duals to each other, i.e., the dual of an order unit [resp. base norm] space is a base norm [resp. order unit] space.

The *state space* of an order unit space (A, e) is defined by $S(A) = \{\omega \in (A^*)^+;\ \|\omega\| = 1\} = \{\omega \in A^*; \|\omega\| = \omega(e) = 1\}$, which abstracts the corresponding definitions for unital C*- and JB-algebras. Then, the *pure state space* $P(A)$ is given by $P(A) = \partial_e S(A)$, and, since $S(A)$ is w*-compact convex, by the Krein-Milman theorem, $S(A) = cl.conv.\,P(A)$. The Kadison's function representation theorem is generalized in this abstract setting as the embedding $A \hookrightarrow A(S(A)) := A(S(A), \mathbf{R})$, where the map is onto if A is complete.

If an order unit space (A, e) has a base norm space (V, K) as its predual, i.e., $(A, e) = (V, K)^*$, then the *normal state space* $N(A) = K$ and $K \cong A^b(K) := A^b(K, \mathbf{R})$ (order isomorphism).

(cf. references [3], [12], [61])

5. Alfsen-Shultz theory.

Let (A, e) be an order unit space which is the dual of a base norm space (V, K), i.e., $A \cong A^b(K)$. A w*-continuous positive projection (idempotent map) $P : A \to A$ with $\|P\| \leq 1$ is said to be a *P-projection* if there exists a w*-continuous positive projection $P' : A \to A$ with $\|P'\| \leq 1$ such that

$$ker^+ P = im^+ P', \quad im^+ P = ker^+ P'$$

$$ker^+ P^* = im^+ P'^*, \quad im^+ P^* = ker^+ P'^*$$

where P' is necessarily unique and called a *quasicomplement* of P. To every P-projection P is associated a *projective unit* $u_P = Pe$ and a *projective face* $F_P = K \cap im\, P^* = \{\omega \in K \,; < u_P, \omega >= 1\}$. The sets of all P-projections on A [resp. projective units in A, projective faces of K] are denoted by $\mathcal{P}(A)$ [resp. $\mathcal{U}(A), \mathcal{F}(A)$]; where an odering is defined by $P \precsim Q$ iff $PQ = QP = P$ in $\mathcal{P}(A)$, $u_P \leq u_Q$ in $\mathcal{U}(A)$ and $F_P \subseteq F_Q$ in $\mathcal{F}(A)$, by which $\mathcal{P}(A), \mathcal{U}(A), \mathcal{F}(A)$ are order isomorphic under the maps $P \leftrightarrow u_P \leftrightarrow F_P$.

If A is a JBW-algebra and V is the predual of A, then the P-projections are exactly the maps $U_p : a \mapsto \{pap\} = 2p \circ (p \circ a) - p \circ a$ for all $a \in A$, where p is an idempotent and $\{pap\}$ is the Jordan triple product; $u_p = U_p e = p$; $F_p = \{\omega \in K \,; < p, \omega >= 1\}$. In particular, if A is the self-adjoint part of a W*-algebra, the map U_p reduces to $U_p a = pap$ $(a \in A)$ with a self-adjoint projection $p \in A$.

Let $(A, e) = (V, K)^*$, then a face F of K is said to be norm exposed if $F = \{\omega \in K \,; < a, \omega >= 0\}$ for some $a \in A^+$. Since the complement of a projective face F_P

is defined by $F_P^\sharp = F_{P'} = \{\omega \in K\,;\, < u_P, \omega > = 0\}$, every projective face is norm exposed. Two elements $a, b \in A^+$ are said to be ***orthogonal*** (denoted by $a \perp b$), if there exists a projective face F_P of K such that $a = 0$ on F_P and $b = 0$ on $F_P^\sharp = F_{P'}$. Then, (A, e) and (V, K) with $A = V^*$ are defined to be in ***spectral duality*** if every norm exposed face of K is projective and every $a \in A$ admits a unique decomposition $a = a^+ - a^-$ such that $a^+, a^- \in A^+$ with $a^+ \perp a^-$. It is shown in [5], [7] that, if (A, e) and (V, K) are in spectral duality, then $\mathcal{P}(A), \mathcal{U}(A), \mathcal{F}(A)$ are complete orthomodular lattices with the complementations $P \mapsto P'$, $u_p \mapsto e - u_P$, $F_P \mapsto F_P^\sharp$, which correspond to each other under the isomorphisms $\mathcal{P}(A) \leftrightarrow \mathcal{U}(A) \leftrightarrow \mathcal{F}(A)$; and every $a \in A$ admits a unique spectral resolution $a = \int \lambda \, de_\lambda$ with $e_\lambda \in \mathcal{U}(A)$ for $\lambda \in \mathbf{R}$, which generalizes the functional calculus for W*-algebras and JBW-algebras developped in [8]. Based on this theory, they obtained the geometric characterizations of the state spaces of JB-algebras [6], and C*-algebras [4], which we summarized in Chapter 7.

(cf. references [4], [5], [6], [7], [8], [45], [67])

6. Nuclear spaces and rigged Hilbert spaces.

Let Ω be a countable normed space equipped with the norms

$$\|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots \leq \|\cdot\|_n \leq \|\cdot\|_{n+1} \leq \cdots .$$

Denoting by Ω_n the completion of Ω with respect to $\|\cdot\|_n$, we have the sequence

$$\Omega \subset \cdots \subset \Omega_{n+1} \subset \Omega_n \subset \cdots \subset \Omega_2 \subset \Omega_1 .$$

Then Ω is defined to be *nuclear* if and only if, for any m, there exists $n > m$ such that the embedding $T_m^n : \Omega_n \to \Omega_m$ is given by a nuclear operator. More generally, let Ω be a (complete) nuclear space with a scalar product which is continuous relative to the nuclear topology, and let H_Ω be the completion of Ω with respect to this scalar product. Then the triplet $\Omega \subset H_\Omega \subset \Omega'$ is called a *rigged Hilbert space* (cf.[39]).

We note that every separable Hilbert space is equipped with a rigged Hilbert space, which is isomorphically given by $\mathcal{S}(\mathbf{R}^k) \subset \mathcal{L}^2(\mathbf{R}^k) \subset \mathcal{S}'(\mathbf{R}^k)$, where $\mathcal{S}(\mathbf{R}^k)$ denotes the space of rapidly decreasing smooth functions on $\mathbf{R}^k$, and the countable norms on $\mathcal{S}(\mathbf{R}^k)$ are given by

$$\|f\|_n = \sup_{|\alpha|,|\beta| \leq n} \|x^\alpha D^\beta f(x)\|_{L^2(\mathbf{R}^k)} \,,$$

where

$$\begin{aligned} x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_k} \qquad && (|\alpha| = \sum_{i=1}^{k} \alpha_i) \\ D^\beta &= \frac{\partial^{\beta_1 + \cdots + \beta_k}}{\partial x_1^{\beta_1} \cdots \partial x_k^{\beta_k}} \qquad && (|\beta| = \sum_{j=1}^{k} \beta_j)\,. \end{aligned}$$

This system of Hilbert norms makes all Ω_n into Hilbert spaces, and Ω is called a *countably Hilbert nuclear space.* Consequently, for any CP-map $\psi = V^* \pi V$ with a representation π on a separable Hilbert space H_π, there exists a countably Hilbert nuclear space Ω such that $\Omega \subset H_\pi \subset \Omega'$ is a rigged Hilbert space.

Let Ω be a countably Hilbert nuclear space. Since the norm of the Hilbert space H_Ω is continuous with respect to the nuclear topology on Ω, there exists $n \in \mathbf{N}$ such that the embedding $T : \Omega \to H_\Omega$ is a nuclear operator with respect to the norm $\| \cdot \|_n$ (cf.[39]). In this setting, I.M. Gelfand and N.Ya. Vilenkin showed ([39;§ 4.4 Th.1']) that, if H_Ω is represented as a direct integral

$$H_\Omega = \int_Z^\oplus H_\Omega(\zeta)\, d\mu(\zeta) \qquad \text{with} \qquad h = \int_Z^\oplus h(\zeta)\, d\mu(\zeta) \quad \text{for } h \in H_\Omega,$$

then, for every $\zeta \in Z$, there exists a nuclear operator $T_\zeta : \Omega \to H_\Omega(\zeta)$ such that, for any $f \in \Omega$,

$$T_\zeta(f) = f(\zeta) \qquad \text{for } \mu - a.e. \zeta \in Z.$$

(cf. references [39], [66])

7. Tomita-Takesaki theory.

A W*-algebra $\mathcal{M} \subset B(\mathcal{H})$, which is realized on a Hilbert space H, is called a von Neumann algebra. In this setting, a non-degenerate *-algebra $\mathcal{M} \subset B(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{M}$ is weakly closed, or equivalently $\mathcal{M} = \mathcal{M}''$. where the commutant of $\mathcal{N} \subset B(\mathcal{H})$ is defined by $\mathcal{N}' = \{a \in B(\mathcal{H}); ab = ba \text{ for all } b \in \mathcal{N}\}$.

For the convenience of our discussion in Chapter 8, we give a brief sketch of the Tomita Takesaki theory for "standard" von Neumann algebras (defined below) following the method of van Dale-Rieffel ([25], [26]). We note that Tomita-Takesaki's theorem is proved for any von Neumann algebra, where the essential ideas of the proofs are parallel.

For a W*-algebra M, the following conditions are equivalent.

(i) M is *σ- finite* , i.e., the cardinarity of all collections of mutually orthogonal projections of M is at most countable.

(ii) M has a *faithful* nomal state ω, i.e., $\omega(a) > 0$ for all nonzero $a \in M^+$.

(iii) M is represented as a von Neumann algebra $\pi(M) \subset B(\mathcal{H})$ which admits a *cyclic and separating vector* $\xi \in \mathcal{H}$, i.e., $[\pi(M)\xi] = \mathcal{H}$ and $\pi(a)\xi = 0$ implies $\pi(a) = 0$.

The representation π of (iii) is called the *standard representation* of M and the von Neumann algebra $\mathcal{M} = \pi(M)$ is called a *standard von Neumann algebra* . Let $\mathcal{M} \subset B(\mathcal{H})$ be a standard von Neumann algebra with a cyclic and separating vector $\xi \in \mathcal{H}$, then $\mathcal{M}'$ is also a standard von Neumann algebra with the same cyclic and separating vector $\xi \in \mathcal{H}$. Define the following antilinear operators on $\mathcal{H}$.

$$S_0 : a\xi \mapsto a^*\xi \quad \text{for } a \in \mathcal{M}$$

$$F_0 : a'\xi \mapsto a'^*\xi \quad \text{for } a' \in \mathcal{M}'.$$

Then S_0 and F_0 are densely defined closable unbounded operators, and the closures $S = \bar{S}_0$, $F = \bar{F}_0$ admit the polar decompositions

$$S = J\Delta^{\frac{1}{2}}, \qquad F = J\Delta^{-\frac{1}{2}}.$$

The positive self-adjoint operator Δ is called the *modular operator* , and the anti-unitary operator J is called the *modular conjugation* associated with the standard pair $(\mathcal{M}, \xi)$.

Then the Tomita-Takesaki theorem states that

$$(1) \qquad J\mathcal{M}J = \mathcal{M}'$$

$$(2) \qquad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M} \quad \text{for all } t \in \mathbf{R}.$$

It follows from the first assertion that $\mathcal{M}$ and $\mathcal{M}'$ are *-anti-isomorphic. Hence, there exists a natural identification of $\mathcal{M}$ with the opposite algebra $\mathcal{M}^{\circ}$ which is defined to be the algebra consisting of the same elements of $\mathcal{M}$ with the reversed multiplication of $\mathcal{M}$, i.e., $(ab)^{\circ} = b^{\circ}a^{\circ}$, which is characterized by the existence of a *-anti-isomorphism from M onto M°. This point of view is used in Chapters 8.

From the second assertion,

$$\sigma_t(a) = \Delta^{it} a\, \Delta^{-it} \quad \text{for } a \in \mathcal{M}$$

defines a σ-weakly continuous one-parameter *-automorphism group on $\mathcal{M}$. Takesaki showed that σ_t is unique under the *KMS-condition* on the faithful normal state ω defined by ξ, i.e., there exists $\beta > 0$, and, for each pair $a, b \in \mathcal{M}$, there exists a holomorphic function $F_{a,b}(z)$ in $D_\beta = \{z \in \mathbf{C}; -\beta < Im\, z < 0\}$ with boundary values $F_{a.b}(t) = \omega(\sigma_t(a)b)$ and $F_{a,b}(t - i\beta) = \omega(b\sigma_t(a))$ for all $t \in \mathbf{R}$.

(cf. references [19], [25], [26], [60], [72], [74], [76])

REFERENCES

1. C.A. Akemann, *The general Stone-Weierstrass problem*, J. Functional Analysis 4 (1969), 277-294.

2. C.A. Akemann and F.W. Shultz, *Perfect C^*-algebras*, Memoirs Amer. Math. Soc. **326** (1985).

3. E.M. Alfsen, "Compact convex sets and boundary integrals," Springer Verlag, Berlin-Heidelberg-New York, 1971.

4. E.M. Alfsen, H. Hanche-Olsen and F.W. Shultz, *State spaces of C^*-algebras*, Acta Math. **144** (1980), 267-305.

5. E.M. Alfsen and F.W. Shultz, *Non-commutative spetral theory for affine function spaces on convex sets*, Memoirs American Math. Soc. **172** (1976).

6. E.M. Alfsen and F.W. Shultz, *State spaces of Jordan algebras*, Acta Math. **140** (1978), 155-190.

7. E.M. Alfsen and F.W. Shultz, *On non-commutative spectral theory and Jordan algebras*, Proc. London Math. Soc. **38** (1979), 497-516.

8. E.M. Alfsen, F.W. Shultz and E. Stϕrmer, *A Gelfand-Neumark theorem for Jordan algebras*, Advaces in Math. **28** (1978), 11-56.

9. J. Anderson and J.W. Bunce, *Stone Weierstrass therem for separable C^*-alg ebras*, J. Operator Theory **6** (1981), 363-374.

10. H. Araki, *Some properties of modular conjugation operator of von Neumann algebras and a non commutative Radon-Nikodym theorem with a chain rule*, Pacific J. Math. **50** (1974), 309-354.

11. W.B. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141-224.

12. L. Asimow and A.J. Ellis, "Convexity theory and its applications in functional analysis," Academic Press, London, 1980.

13. J. Bellissard and B. Iochum, *Homogeneous self-dual cones, versus Jordan alge-*

bras. The theory revisited., Ann. Inst. Fourier, Grenoble 28, **1** (1978), 27-67.

14. J. Bellissard and B. Iochum, *Spectral theory for facially homogeneous symmetric self-dual cones*, Math. Scand. **45** (1979), 118-126.

15. J. Bellissard and B. Iochum, *L'algèbre de Jordan d'un cône autopolaire facialement homogène*, C.R. Acad. Sci. Paris, Ser. A **288** (1979), 229-232.

16. J. Bellissard, B. Iochum and R. Lima, *Homogeneous and facially homogeneous self-dual cones*, Linear Algebra and Appl. **19** (1978), 1-16.

17. K. Bichteler, *A generalization to the non-separable case of Takesaki's duality theorem for C^*-algebras*, Inventiones Math. **9** (1969), 89-98.

18. C. Blondia, *Integration in locally convex spaces*, Simon Stevin, A Quarterly Journal of Pure and Applied Mathematics **55** (1981), 81-102.

19. O. Bratteli and D.W. Robinson, "Operator Algebras and Quantum Statistical Mechanics I," Springer Verlag, New York-Heidelberg-Berlin, 1979.

20. J.K. Brooks and N. Dinculeanu, *Lebesgue-type spaces for vector integration, linear operators, weak completeness and weak compactness*, J. Math. Anal. Appl. **54** (1976), 348-389.

21. J.W. Bunce, *Approximating maps and a Stone-Weierstrass theorem for C^*-alg ebras*, Proc. Amer. Math. Soc. **79** (1980), 559-563.

22. J.W. Bunce, *The general Stone-Weierstrass problem and extremal completely positive maps*, Manuscripta Math. **56** (1986), 343-351.

23. M-D Choi, *A Schwarz inequality for positive linear maps on C^*-algebras*, Illinois J. Math. **18** (1974), 565-574.

24. A. Connes, *Caractérisation des espaces vectoriels ordonnés sous-jacents aux algébras de von Neumann*, Ann. Inst. Fourier, Grenoble 24, **4** (1974), 121-155.

25. A. van Dale, *A new approach to Tomita-Takesaki theory of generalized Hilbert algebras*, J. Func. Anal. **15** (1974), 378-393.

26. A. van Dale and M. Rieffel, *A bouded operator approach to Tomita-Takesaki theory*, Pac. J. Math. **69** (1977), 187-221.

27. J. Diestel and J.J. Uhl, Jr., "Vector Measures," Mathematical Surveys No.15, AMS, Providence, 1977.

28. J. Dixmier, "C*-algebras," (English edition), North-Holland, Amsterdam-New York-Oxford, 1977.

29. J. Dixmier, "Von Neumann Algebras," (English edition), North-Holland, Amsterdam-New York-Oxford, 1981.

30. E.G. Effros, *Order ideals in a C^*-algebra and its dual*, Duke Math. J. **30** (1963), 391-412.

31. E.G. Effros, *Injectives and tensor products for convex sets and C^*-algebras*, Lecture notes of NATO Institute on "Facial structure of compact convex sets" at the University College of Swansea, Wales, July, 1972.

32. G.G. Emch, "Algebraic methods in statistical mechanics and quantum field theory," John Wiley & Sons, Inc., New York, 1972.

33. G.G. Emch, "Mathematical and conceptual foundations of 20th-century physics," North-Holland, Amsterdam-New York-Oxford, 1984.

34. D.E. Evans and J.T. Lewis, *Dilations of irredusible evolutions in algebraic quantum theory*, Communications of the Dublin Institute for Advanced Studies, Ser.A, No.24, 1977.

35. I. Fujimoto, *The general Stone-Weierstrass problem in convexity theory*, Master's thesis, Hiroshima University, 1982.

36. I. Fujimoto, *On the dual spaces of JBW-algebras*, preprint, 1983.

37. I. Fujimoto, *CP-duality for C^*- and W^*-algebras*, Proc. Symposia in Pure Math. **51**, Operator Theory/Operator Algebras and Applications, 1990, to appear.

38. I. Fujimoto and S. Takahasi, *Equivalent conditions for the general Stone-Wei erstrass problem*, Manuscripta Math. **53** (1985), 217-224.

39. I.M. Gelfand and N. Ya. Vilenkin, "Generalized Functions IV," Applications of Harmonic Analysis, Academic Press, New York-London, 1964.

40. J. Glimm, *A Stone-Weierstrass theorem for C^*-algebras*, Ann. of Math. **72** (1960), 216-244.

41. U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. **37** (1975), 271-283.

42. U. Haagerup and H. Hanche-Olsen, *Tomita-Takesaki theory for JBW-algebras*,

J. Operator Theory **11** (1984), 343-364.

43. H. Hanche-Olsen, *Split faces and ideal structure of operator algebras*, Math. Scand. **48** (1981), 137-144.

44. H. Hanche-Olsen, *On the structure and tensor products of JC-algebras*, Can. J. Math. **35(6)** (1983), 1059-1074.

45. H. Hanche-Olsen and E. Stϕrmer, "Jordan Operator Algebras," Pitman Publ., Boston-London-Melbourne, 1984.

46. B. Iochum, "Cônes autopolaires et algébres de Jordan," Lecture Notes in Mathematics, 1049, Springer Verlag, New York-Heiderberg-Berlin, 1984.

47. B. Iochum and F.W. Shultz, *Normal State spaces of Jordan and von Neumann Algebras*, J. Functional Analysis **50** (1983), 317-328.

48. H.E. Jensen, *Scattered C^*-algebras*, Math. Scand. **41** (1977), 308-314.

49. H.E. Jensen, *Scattered C^*-algebras II*, Math. Scand. **43** (1978), 308-310.

50. R.V. Kadison, *A representation theory for commutative topological algebra*, Memoires Amer. Math. Soc. **7** (1951).

51. R.V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algebras*, Ann. of Math. **56** (1952), 494-503.

52. R.V. Kadison and J.R. Ringrose, "Fundamentals of the theory of operator algebras," vol.1, Elementary theory, Academic Press, New York-London, 1983.

53. R.V. Kadison and J.R. Ringrose, "Fundamentals of the theory of operator algebras," vol.2, Advanced theory, Academic Press, Orlando-London, 1986.

54. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219-255.

55. K. Kraus, *General state changes in quantum theory*, Ann. of Physics **64** (1971), 311-335.

56. G. Lindblad, *On the generators of quantum dynamical semigroups*, Comm. Math. Phys. **48** (1976), 119-130.

57. R. Longo, *Solution of the factorial Stone-Weierstrass conjecture. An application of the theory of standard split W^*-inclusions*, Invent. Math. **76** (1984), 145-155.

58. T. Ogasawara, *Topologies of operator algebras*, J. Hiroshima Univ., Ser. A **19** (1955), 255-272.

59. V.I. Paulsen, "Completely bounded maps and dilations," Longman Scientific & Technical, Essex, 1986.

60. G.K. Pedersen, "C*-algebras and their automorphism groups," Academic Press, London, 1979.

61. R. Phelps, "Lectures on Choquet's theorem," Von Nostrand, Princeton, 1966.

62. S. Popa, *Semi-regular maximal abelian *-subalgebras and the solution to the factor state Stone-Weierstrass problem*, Invent. Math. **76** (1984), 157-161.

63. R.T. Prosser, *On the ideal structure of operator algebras*, Memoirs Amer. Math. Soc. **45** (1963).

64. S. Sakai, *On the Stone-Weierstrass theorem of C^*-algebras*, Tohoku Math. J. **22** (1970), 191-199.

65. S. Sakai, "C*- and W*-algebras," Springer-Verlag, Berlin-Heidelberg-New York, 1971.

66. H.H. Schaefer, "Topological Vector Spaces," Springer Verlag, New York-Heidel berg-Berlin, 1970.

67. F.W. Shultz, *On normed Jordan algebras which are Banach dual spaces*, J. Functional Analysis **31** (1979), 360-376.

68. F.W. Shultz, *Dual maps of Jordan homomorphisms and *-homomorphisms between C^*-algebras*, Pacific J. Math. **93** (1981), 435-441.

69. F.W. Shultz, *Pure states as a dual objects for C^*-algebras*, Commun. Math. Phys. **82** (1982), 497-509.

70. P.J. Stacey, *Locally orientable JBW-algebras of complex type*, Quart. J. Math. Oxford (2) **33** (1982), 247-251.

71. W.F. Stinespring, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211-216.

72. S. Strătilă and L. Zsidó, "Lectures on von Neumann algebras," (English edition), Abacus Press, Tunbridge Wells, Kent, 1979.

73. M. Takesaki, *A duality in the representation theory of C^*-algebras*, Ann. of Math. **85** (1967), 370-382.

74. M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications*, Springer Lecture Notes in Math. 128, 1970.

75. M. Takesaki, "Theory of operator algebras I," Springer-Verlag, New York-Heidelberg-Berlin, 1979.

76. M. Takesaki, "Structure of Operator Algebras," Iwanami, Tokyo, 1983.

77. G.E. Thomas, *Integration of functions with values in locally convex Suslin spaces*, Trans. Amer. Math. Soc. **212** (1975), 61-81.

BIOGRAPHICAL SKETCH

Ichiro Fujimoto was born on October 1, 1951 in Japan. He attended the University of Tokyo from 1971 to 1975, and graduated with a B.S. in physics. He stidied mathematics at Kyoto University (1978-1980) and received the M.S. in mathematics from Hiroshima University in 1982, where he majored in functional analysis under the direction of Professor Fumi-Yuki Maeda. After being a research student at Hiroshima University, he entered the Ph.D. program at the University of Rochester in 1985 to work on operator algebras and applications to mathematical physics under the supervision of Professor Gérard G. Emch. In 1986, he transfered to the University of Florida with his advisor.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Gérard G. Emch, Chairman
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Stephen J. Summers

Stephen J. Summers
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Joseph Glover

Joseph Glover
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentataion and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Bruce H. Edwards
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Pierre Sikivie
Associate Professor of Physics

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August 1990

Dean, Graduate School